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PRINCIPLES OF TRANSMISSION IN TELEPHONY

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PRINCIPLES OF TRANSMISSION IN TELEPHONY

BY

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PREFACE

The present textbook represents an effort to place before the students in electrical engineering, particularly those interested in the fascinating field of telephone engineering, a rigorous mathematical analysis of problems that are actually met in modern telephone practice.

The fundamental ideas involved in electrical transmission are the same whether the application is telephony or heavy power transmission. The detailed development of the theory, however, differs greatly in the two cases on account of the differences in voltage, in frequency and in variation of frequency, in energy values and in length of line. A careful study of these details will furnish not only a firm basis for understanding the more serious problems of telephone transmission, but will strengthen the understanding of the underlying principles of all forms of electrical transmission.

Practical applications of the solutions obtained in the analytical investigations throughout the book are illustrated by fully solved numerical examples, based on the electrical constants of a No. 10 B. & S. aerial line with a sending-end effective difference of potential of 50 volts, at the average telephonic frequency of 796 cycles per second. To impress upon the student the significance of the relationships developed in the various analyses, he should be required to solve additional problems for the same and other lines under conditions of frequency, sending and receiving-end voltages other than those stated above. Table 1 gives for this purpose a list of cable and aerial lines generally used in telephone transmission. The transmission constants of these lines as given in this table are for a frequency of 796 cycles per second.

At the request of his students, the author has omitted in this book the use of the dot under or a vinculum over symbols that

represent vector quantities. The same symbol may designate either a vector quantity or its tensor value. When used as tensors such symbols will generally be found associated with the exponential operator $\epsilon^{j\theta}$ to indicate the vector angular position. The omission of the dot or vinculum from vector symbols simplifies considerably expressions comprising a large number of vector quantities.

The exponential operator $\epsilon^{j\theta}$ to designate the angular position of a vector is used throughout the book in preference to the more generally used polar symbol $\lfloor \underline{\theta}$. This was also done at the suggestion of the author's students. The former notation appealed to them because it is subject to all the laws applicable to exponential functions.

Within the last few years there have appeared a few excellent books and a considerable number of technical papers by prominent men connected with the profession, dealing with various phases of telephone engineering. A more or less complete list of such books and papers is given in the Bibliography. A few of these, available to the author, have been consulted for suggestive material or methods of analysis. The author takes pleasure in acknowledging his indebtedness to these works, particularly to Dr. A. E. Kennelly's *Hyperbolic Functions Applied to Electrical Engineering Problems*. In this work, Dr. Kennelly has given to the profession a mode of analysis, the results of which lend themselves with equal facility to physical interpretation and numerical calculation.

The author takes this opportunity to thank his friend and former student Mr. L. P. Van Houten for his many valuable suggestions and his colleague Professor A. C. Lanier for his kind and willing criticisms; he is under obligation to Mr. B. D. Hull, General Transmission and Protection Engineer of the Southwestern Bell Telephone Company for the "Memoranda" on the Bell Transmission Unit, and to Mr. L. L. Lucking, Division Plant Engineer of the Southwestern Bell Telephone Company for valuable suggestions embodied in Chapter XVII.

In conclusion, the author takes pleasure in expressing his indebtedness to Professor C. E. Tucker of the Massachusetts Institute of Technology for his aid in proofreading, to the publishers for the excellent workmanship of the publication, and particularly to

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M. P. Weinbach.

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CONTENTS

PAGE

CHAPTER I. INTRODUCTION 1

CHAPTER II. HARMONIC FUNCTIONS 5
1. The Trigonometric Expression of the Harmonic Function . . 5
2. Analysis of a Complex Wave into its Harmonic Components . 7
3. The Complex Number and its Relation to the Harmonic Function 9
4. Exponential Equivalents of Trigonometric Functions 11
5. Vector Operations 12
6. Addition and Subtraction 12
7. Multiplication and Division 13
8. Involution and Evolution 13
9. Differentiation and Integration of Vector Quantities 13
10. Trigonometric Relations 14

CHAPTER III. HYPERBOLIC FUNCTIONS 15
11. Analogy between the Hyperbolic and Circular Trigonometric Functions 15
12. Relations between Hyperbolic Functions 18
13. The Exponential Equivalent of the Hyperbolic Functions . . 19
14. Relationship between Circular Trigonometric and Hyperbolic Functions 20
15. Trigonometric and Hyperbolic Functions of Complex Quantities 21
16. Exponential Equivalents of Circular Trigonometric and Hyperbolic Functions of Complex Numbers 23

CHAPTER IV. GENERAL EQUATIONS OF ELECTRIC WAVE PROPAGATION 25
17. Fundamental Relations 25
18. Mathematical Analysis 26
19. Physical Interpretation of the General Transmission Equations . 31
20. The Propagation Constant 32
21. The Attentuation Constant 33
22. The Wave-Length Constant 34
23. Velocity of Propagation 35
24. Scalar Values of the Attenuation and Wave-Length Constants . 35
25. Characteristic Line Impedance. Line Phase Angle. Surge Impedance 36
26. Calculation of the Transmission Constants of No. 10 B. & S. Aerial Line 38
27. Transmission Constants of Telephone Lines 39

PAGE

28. Fundamental Data for Calculating Voltage and Current Values for the No. 10 B. & S. Line under any Receiving-End Conditions 39

CHAPTER V. THE INFINITELY LONG LINE 43
29. Line Impedance and Sending-End Current 43
30. Voltage and Current at any Point along an Infinite Line . . . 44

CHAPTER VI. THE OPEN-CIRCUITED LINE 47
31. Sending-End Impedance 47
32. Sending-End Current 48
33. Voltage at the Open-Circuited End 48
34. Resonance Effects and Natural Line Frequency 50
35. Equivalent Line Inductance and Capacitance 54
36. The Ferranti Effect 56
37. Numerical Illustration 58
38. Voltage at any Point along an Open-Circuited Line 59
39. Variation of Current Values along an Open-Circuited Line . . 62
40. Variation of Impedance along an Open-Circuited Line . . . 65
41. Variation of the Phase Angle between Voltage and Current along an Open-Circuited Line 69
42. Phase Angle at the Open End of a Line 71
43. Points along an Open-Circuited Line at which Voltage and Current are in Phase 72
44. Points along an Open-Circuited Line at which the Phase Angle is Equal to the Characteristic Phase Angle 75

CHAPTER VII. THE SHORT-CIRCUITED LINE 77
45. Sending-End Impedance of a Short-Circuited Line 77
46. Sending-End Current 78
47. Current at the Short-Circuited End 79
48. Voltage at any Point along a Short-Circuited Line 80
49. Current at any Point along a Short-Circuited Line 81
50. Impedance at any Point along a Short-Circuited Line 84
51. Variation of Phase Angle 86

CHAPTER VIII. LINE WITH END-IMPEDANCE $Z_r\epsilon^{j\theta_r}$ 90
52. Sending-End Impedance 90
53. Equivalence between an Infinitely Long Line and the same Line of any Length Loaded with an Impedance Equal to the Characteristic Line Impedance 92
54. Sending-End Current 93
55. Receiving-End Voltage 94
56. Receiving-End Current 94
57. Voltage at any Point along a Line with End-Impedance . . . 95
58. Current at any Point along a Line with End-Impedance . . . 95
59. Impedance at any Point along a Line with End-Impedance . . 95
60. The Composite Line 96

PAGE
61. Sending-End Impedance of a Composite Line 96
62. Voltage and Current at the Termini and Junctions of a Composite Line 97
CHAPTER IX. ENERGY TRANSFER AND REFLECTION PHENOMENA . . 101
63. Physical Conception of Electromagnetic Waves 101
64. Mechanism of Energy Transfer 101
65. Standing Waves 102
66. Traveling Waves 103
67. Reflection of Electromagnetic Waves 105
68. Reflections on Open Circuit 105
69. Reflections on Short Circuit 108
70. Reflections at the End of a Line Loaded with an Impedance Z_r; Voltage Transition and Reflection Coefficients . . . 110
71. Current Transition and Reflection Coefficients 113
CHAPTER X. POWER RELATIONS 114
72. The Infinite Line 114
73. Line with End Impedance $Z_r e^{j\theta_r}$ 116
74. Variation of Receiver Voltamperes $(VA)_r$ with the Modulus of Receiver Impedance 116
75. Variation of Voltamperes with the Modulus of Receiver Impedance if $S = n\lambda$ and $\theta_r = \alpha$ 117
76. Variation of Voltamperes with the Modulus of the Receiver Impedance when $S \neq n\lambda/4$ and $\theta \neq \alpha$ 119
77. Receiver Impedance for Maximum Voltamperes. Terminal Gain and Loss 122
78. Variation of Receiver Voltamperes with Receiver Phase Angle . 129
79. Voltampere Transition and Reflection Coefficients 132
80. Voltampere Transition and Reflection Coefficients when $Z_r = Z_0 \tanh pS$ 134
81. Watts Transition and Loss Coefficients 136
CHAPTER XI. THE EQUIVALENT CIRCUIT 139
82. Definition of the Equivalent Circuit 139
83. Nominal and Equivalent Circuits of a Transmission Line . . . 139
84. The Nominal T Circuit 142
85. The Equivalent T Circuit 143
86. Degree of Equivalence between the Nominal T Circuit and the Line it Represents 147
87. Mathematical and Actual Equivalent T Circuits 147
88. Equivalent T Circuits for Lines with Definite End Conditions. The Open-Circuited Line 150
89. The Equivalent T Circuit of a Short-Circuited Line 152
90. The Equivalent T Circuit of a Line with any End Impedance $Z_r e^{j\theta_r}$ 153
91. The Nominal Π Circuit 155
92. The Equivalent Π Circuit 156
93. Degree of Equivalence between the Nominal Π Circuit and the Line it is Supposed to Represent 159

PAGE

94. The Equivalent Π Circuit of an Open-Circuited Line 160
95. Equivalent Π Circuit of a Short-Circuited Line. 161
96. The Equivalent Π Circuit of a Line with any End Impedance $Z_r\epsilon^{j\theta_r}$ 162

CHAPTER XII. THE PHENOMENA OF DISTORTION; DISTORTIONLESS TRANSMISSION 163

97. Variation of the Characteristic Line Impedance with Frequency. Aerial Lines 163
98. Variation of the Characteristic Impedance with Frequency when Leakage Conductance is Neglected 165
99. Variation of Characteristic Impedance with Frequency when G and L are both Neglected 166
100. Causes of Wave Distortion 167
101. Variation of the Propagation Constant with Frequency. Aerial Lines 167
102. Variation of the Propagation Constant with Frequency for Lines with Negligible Leakage Conductance 169
103. Variation of the Propagation Constant with Frequency for Lines with Negligible Leakage and Inductance 170
104. Conditions for Distortionless Transmission of Telephonic Waves 171
105. Methods of Varying the Line Constants. Variation of RC . . 174
106. Variation of LG 175

CHAPTER XIII. THE INDUCTIVELY LOADED LINE 178

107. Pupin System of Loading 178
108. The Equivalent Circuit of the Regularly Series-Loaded Line . . 179
109. The Uniform Line Equivalent to the Regularly Series-Loaded Line 180
110. Characteristic Impedance of the Regularly Series-Loaded Line . 182
111. Determination of the Propagation, Attenuation, and Wave-Length Constants of the Regularly Series-Loaded Line 182
112. The Uniform Line, Nominal to the Regularly Series-Loaded Line 184
113. Propagation Constant of the Line Nominal to the Loaded Line . 185
114. Degree of Equivalence between the Smooth Line Nominal to the Loaded Line, and the Loaded Line 186
115. Minimum Number of Loading Coils per Wave-Length. Maximum Spacing Distance between Loading Coils 188
116. Calculation of the Spacing Distance between Loading Coils . . 190
117. Calculation of the Transmission Constants of the Line Nominal to the No. 10 B. & S. Regularly Series Loaded Line . . . 191
118. Calculation of the Transmission Constants of the Uniform Line Equivalent to the Regularly Series-Loaded No. 10 B. & S. Line 193
119. Calculation of the Transmission Constants of the Ideally Loaded No. 10 B. & S. Line 195
120. Effect of Artificial Loading on Energy Transfer 196
121. Use of Terminal Repeating Coils for the Apparent Equilization of Impedances 198
122. Loading of Underground Cable Lines 201

PAGE

CHAPTER XIV. AMPLIFIERS 203
123. Amplification, a Necessity for Long-Distance Transmission . . 203
124. The Shreeve Repeater 203
125. Fundamentals of Thermionic Emission. The Thermionic Tube . 204
126. The Two-Electrode Thermionic Tube 206
127. The Three-Electrode Thermionic Tube 209
128. Amplifier Action of the Three-Electrode Tube 211
129. Power Amplification 216
130. Conditions for Maximum Power Amplification 217
131. The Standard Amplifying Tube 217
132. The Thermionic Tube as an Oscillator 217
133. The Oscillator on Load. Operating Conditions of the Thermionic Oscillator 220

CHAPTER XV. REPEATER OR AMPLIFYING CIRCUITS 225
134. One-Way Amplification 225
135. Two-Way, One-Element, Amplifying Circuits 227
136. Two-Way, One-Element, Parallel-Series Amplification . . . 231
137. Two-Way Element, Amplifying Circuits 233

CHAPTER XVI. LINE IMPEDANCE AND ITS MEASUREMENT 235
138. Variation of Line Impedance with Frequency 235
139. Effect of Irregularities on the Impedance-Frequency Curve of a Line 238
140. Measurement of Line Impedance by Open-Circuit and Short-Circuit Tests 241
141. Calculation of the Characteristic Impedance and the Characteristic Phase Angle from Test Data 242
142. Calculation of the Propagation Constant, the Attenuation Constant, and the Wave-Length Constant from Test Data . . 244
143. Calculation of R, L, C, and G from Test Data 246
144. Location of Irregularity. Theoretical Considerations . . . 247
145. Line Impedance Measurements. Practical Considerations . . 248
146. The Impedance Bridge 250
147. Balancing Terminal Networks 255

CHAPTER XVII. TRANSMISSION EQUIVALENTS 257
148. The Standard Cable; the Standard Mile 257
149. Transmission Equivalents of Lines Free of Terminal Reflections . 258
150. Nominal Line Admittance as a Measure of the Transmitting Ability of a Line 261
151. Transmission Equivalents of Lines in Terms of the Standard Cable Free of Terminal Reflections under Similar Transmitting-End Conditions 262
152. Transmission Equivalents of Lines in Terms of Standard Cable with Definite Sending-End and Receiving-End Terminations 266
153. Measurement of Telephonic Transmission Equivalents . . . 269
154. The Transmission Measuring Unit 269
155. Calibration of the Transmission Measuring Unit 272

PAGE

156. Measurement of the Telephonic Transmission Equivalent . . 272
157. Measurement of the Telephonic Value of a Line 274
158. The Loop Test 276
159. Disadvantages of the Standard Mile as a Transmission Unit. A Suggested New Transmission Unit 277
160. Proposed A. T. & T. Transmission Unit 279
161. Discussion of the Proposed New A. T. & T. Unit 283

Bibliography 287

List of Symbols 291

Index 299

PRINCIPLES OF TRANSMISSION IN TELEPHONY

PRINCIPLES OF TRANSMISSION IN TELEPHONY

CHAPTER I

INTRODUCTION

The essentials of electrical speech transmission are: (1) the conversion of sound waves produced by articulated speech into electric waves by means of the transmitter; (2) the propagation of these electric waves along a transmission line; and (3) their conversion into sound waves at the receiving end of the line through the agency of the receiver.

The rapid changes in air pressure constituting sound waves, produced in front of a telephone transmitter, will cause the transmitter diaphragm to vibrate with a frequency equal to, and with an amplitude nearly proportional to, that of the sound waves.

The transmitter consists of a carbon microphone which is usually associated with an induction coil. In ordinary local battery arrangements, the transmitter, primary winding of the induction coil, and a battery are connected in series in a local circuit. The secondary winding of the induction coil is connected to the transmission line and receiver in series. In central battery service the transmitter and primary winding are commonly connected in series to the line, while the secondary winding is connected in a branched local circuit in series with the receiver and a condenser.

The microphone is a circular shallow box, filled with granulated carbon constituting a variable resistance, subject to the vibrating action of a flexible diaphragm, which is in front of, and insulated from, the granulated carbon resistance. The change in resistance following the vibrations of the diaphragm will cause changes in the primary current values. This causes secondary electromotive forces and currents, which form an electro-magnetic train of waves of somewhat the same shape as, and of the same frequencies as, the train of sound waves producing it. These waves travel at a definite velocity toward the receiving end of the line. On reaching the

receiver, the energy transferred through the electro-magnetic waves is spent in causing a vibratory action on a flexible diaphragm in the receiving apparatus. This causes rapid changes in air pressure, thus reproducing sound waves closely similar to those originally impressed upon the transmitter diaphragm. By means of a variety of devices, the reproduced sound in front of the receiver may be made an almost exact counterpart of the sounds impressed upon the transmitter, with respect to pitch, loudness, and quality or "timbre."

Since the pitch is determined by the frequency, *i.e.*, the rate of vibration of the body producing the sound waves, and the loudness of the sound is determined by the amplitude, and since the quality or timbre depends upon the overtones or harmonics composing the sound waves, it is easy to see that, to obtain an exact reproduction of the sound waves, the frequency of the component harmonics, their amplitude, and their relation to one another, *i.e.*, the wave shape, must be maintained in the conversion from sound to electric waves, during the propagation of these waves, and in their reconversion into sound waves at the receiving end of the line.

It must be kept in mind, however, that these essentials of perfect speech transmission are never quite realized in practical telephony. Distortion of the original wave shape, and attenuation of the component harmonics of the electric train of waves, are always present in greater or lesser degree. When no corrective means are employed, this causes indistinctness of the reproduced sounds.

Sound waves are produced by the motion of air particles to and fro in the direction in which the sound is transmitted. When the sound is a musical one or a prolonged vowel sound, the motion of the air particles is a repeating function of time, of the type called *harmonic*, generally referred to as a ***sine function*** of time. When the sound is consonantal or sibilant in character, the wave produced is irregular and non-repeating, and as a rule more difficult to convert and transmit electrically in comparison with musical or vowel sounds. Hence, since articulated speech is composed of a variety of superimposed vowel and consonantal sounds, sound waves will be very irregular in shape. Such a wave, however, has a definite average frequency or period of vibration, a definite average wave-length (which determines the pitch), and a definite average amplitude (which determines the loudness). The actual wave shape, obviously, depends on the relative amplitude of the superimposed

harmonics or overtones and on the relative phase displacements of these component harmonics with respect to one another.

Since any wave, no matter how irregular in shape, may be considered as the resultant of superimposed pure harmonics or sine curves of gradually decreasing amplitudes and periods of vibration, the mode of propagation of such a wave may be obtained through a study of the mode of propagation of its component harmonics. The resolving of a complex wave into its constituent harmonics may be accomplished with comparative facility by either a direct application of Fourier's series, or by any other of the several known methods of approximation.

It is found in practical telephony that the sound wave of even a pure tone, as reproduced at the receiving end of a line, will lag behind, or be retarded in phase, with respect to the original wave impressed at the sending end. This retardation depends largely upon the physical characteristic of the sending and receiving apparatus (*i.e.* the inertia and elasticity of transmitter and receiver diaphragms, the inductance and capacitance of the electrical circuit comprised in it), the various apparatus introduced into the circuit, the electrical characteristics of the line over which the waves are transmitted, and the presence or absence of reflected waves.

Since all these factors are affected by the frequency, the phase relationship between the various component harmonics of a sound wave, as reproduced by the receiver, will naturally differ from the phase relationship of the component harmonics of the original sound wave impressed upon the transmitter. Hence the received sounds will differ in character from those sent, unless corrective means, such as artificial inductive loading of the line, are employed. In addition to this, due to the dissipative loss during propagation, as evidenced by the attenuation or gradual decrease in the amplitude of the component harmonics of the wave, the reproduced sounds may be quite weak in comparison with those impressed upon the transmitter, unless the energy loss is compensated for or resupplied to the line by means of *amplifiers* or *repeaters*, from a source external to the line itself.

The attenuation of a pure sine wave of a definite frequency can be easily calculated. Since a complex telephonic wave is the resultant of a number of sine waves of different frequencies and amplitudes, traveling over the same circuit, the average attenuation,

resulting in the weakening of the reproduced sounds, can also be calculated.

The usefulness of mathematical analysis of the physical problems involved in electric speech-transmission is very apparent. Such analysis furnishes the means of ascertaining the mode of propagation of the waves, the distortion and attenuation that take place during propagation, the relationship between the factors that are responsible for the distortion and attenuation, and the means that may be utilized to eliminate or at least minimize them.

The mathematical processes most useful in the analysis of these problems comprise a knowledge of the properties of trigonometric functions including Fourier's series, vector analysis and the algebraic expression of vectors by means of complex numbers, the exponential equivalent of circular trigonometric functions, hyperbolic functions and their exponential equivalents, and the interrelationship between hyperbolic and circular trigonometric functions.

CHAPTER II

HARMONIC FUNCTIONS

1. The Trigonometric Expression of the Harmonic Function. — We shall have to consider regular repeating functions that can be expressed by an equation of the form

$$x = X \sin \theta, \tag{1}$$

or of the form

$$x = X \cos \phi, \tag{2}$$

where x is the instantaneous value of the repeating or varying quantity, X its maximum value or "amplitude," and θ or ϕ the "argument" or angular displacement from a definite point of reference. The graph representing the variation of the quantity x is called a ***sine curve,*** or a ***cosine curve,*** respectively.

Since a sine or a cosine curve will be described by the right line projection of a line, called a ***vector,*** of uniform length equal to the amplitude of the sine curve, rotating about a point at a uniform angular velocity ω, the equation (1) may be written in the form

$$x = X \sin \omega t, \tag{3}$$

where ωt is the conventional counter-clockwise angular displacement of the rotating vector in time t from the positive horizontal axis of reference.

It is evident that, for a complete revolution, each point of the rotating vector will describe a circle, *i.e.*, each point will pass over 2π radians, while the projection of the vector will describe a sine curve. In one second the rotating vector will pass over ω radians, and its projection will describe $\omega/(2\pi)$ complete sine curves per second. The number of complete sine curves, or ***cycles,*** thus described by the projection of the rotating vector per second, is called the ***frequency,*** and is usually denoted by the symbol f. It follows therefore that

$$f = \frac{\omega}{2\pi}. \tag{4}$$

The time required by the projection of the rotating vector to describe one cycle is called the *period;* its value is

$$T = \frac{1}{f}. \tag{5}$$

From what precedes it follows that

$$\theta = \omega t = 2\pi ft, \tag{6}$$

and equation (1) may then be written in the form

$$x = X \sin 2\pi ft. \tag{7}$$

Any electromotive force or current that is a regular repeating function of time, following the pure sine law, may be represented by any of the following equations, respectively,

$$e = E \sin \theta = E \sin \omega t = E \sin 2\pi ft, \tag{8}$$

$$i = I \sin \theta = I \sin \omega t = I \sin 2\pi ft. \tag{9}$$

If the current and the electromotive force producing it pass through zero and have their respective maxima at the same instant of time, as is the case with pure resistance circuits, the two are said to be *in phase*, and their mathematical representation with respect to one another is as shown by equations (8) and (9). The type of circuit may, however, force the current either to lag behind the voltage producing it, as with an inductive circuit, or to lead the voltage, as is the case with a dielectric circuit. The voltage and current are then said to be *out of phase.* If ϕ is the phase angle, the current equation with respect to that of the voltage may be written in the form

$$i = I \sin (\omega t \pm \phi), \tag{10}$$

where the + sign indicates that the current is leading the voltage, and the − sign that the current is lagging behind the voltage.

The value of an alternating current (or voltage) that will produce the same heating effect as a direct current (or voltage) is called ***effective value.*** It is well known that this value is the *root mean square* of the instantaneous values. In terms of the maximum current, the effective value of the current may be obtained from the relationship

$$I_f = I_m \frac{1}{2\pi} \sqrt{\int_0^{2\pi} \sin^2 \theta \, d\theta} = \frac{I_m}{\sqrt{2}} = .707 \, I_m$$

By definition, the effective value is not a vector. Nevertheless, since its numerical value is a definite fraction of the maximum,

which is a vector quantity, the effective value is conventionally considered as a vector, so as to avoid the solving of problems in terms of maxima and calculating the effective values therefrom.

The average or the arithmetic mean of the instantaneous values of a harmonically varying alternating current (or electromotive force) is .636 I_m, and the ratio of the effective to the average value is accordingly 1.11. This ratio is called ***form factor***. Any deviation from this value is an indication that the repeating function is not regular; in other words, the current or the voltage does not follow the pure sine law.

2. Analysis of a Complex Wave into its Harmonic Components. — Any repeating function, no matter how irregular in shape it may be, may be thought of as the resultant of a definite number of harmonically repeating functions of different amplitudes and periods of vibration, superimposed on one another either in phase or out of phase with one another. The frequencies of these harmonics may be any positive number of times the frequency of the fundamental, *i.e.*, of that harmonic whose frequency is equal to that of the wave proper. Where the harmonic frequencies are integral multiples of the fundamental, the general equation of a complex wave may be written in the following form, in which y represents any instantaneous value,

$$(11)\quad y = A_0 + A_1 \sin\theta + A_2 \sin 2\theta + A_3 \sin 3\theta + \cdots + A_n \sin n\theta + B_1 \cos\theta + B_2 \cos 2\theta + B_3 \cos 3\theta + \cdots + B_n \cos n\theta.$$

The analytical problem consists in the determination of the numerical values and the signs of the amplitudes of each component harmonic.

The integration method of determining these amplitudes consists in converting each component member of the above equation for a complex wave into an integral with definite limits. In carrying out the process of integration, all except the member whose constant is to be determined, vanishes. Thus, multiplying each term of equation (11) by $\int_0^{2\pi} d\theta$, and carrying out the integration for all terms, we see that all terms except the first vanish, and that the expression is reduced to the form

$$\int_0^{2\pi} y\, d\theta = \int_0^{2\pi} A_0\, d\theta$$

or

$$2\pi A_0 = \int_0^{2\pi} y\, d\theta$$

or

$$A_0 = \frac{1}{2\pi}\int_0^{2\pi} y\, d\theta \tag{12}$$

Since the integral on the right of the equality sign in the last equation is the area under the complex curve for one complete cycle, it is evident that A_0 is the arithmetic average of the instantaneous values of the curve itself, and may then be obtained very easily by adding the instantaneous values of y and dividing by their number. Clearly, the larger the number of instantaneous values, the more accurate will be the value of A_0.

To calculate the amplitude of the second sine harmonic, multiply every term of equation (11) by $\int_0^{2\pi} \sin 2\theta\, d\theta$. Carrying out the integration, we see that all the integral terms, with the exception of the one whose amplitude is to be obtained, will vanish, and the expression is reduced to the form

$$\int_0^{2\pi} y \sin 2\theta\, d\theta = A_2 \int_0^{2\pi} \sin^2 2\theta\, d\theta = A_2 \int_0^{2\pi} \frac{1 - \cos 4\theta}{2}\, d\theta,$$

whence

$$A_2 = \frac{1}{\pi}\int_0^{2\pi} y \sin 2\theta\, d\theta. \tag{12}$$

This means that the amplitude A_2 of the second sine harmonic is numerically equal to the average instantaneous value of the curve whose equation is $y \sin 2\theta$. In other words, to obtain the numerical value of A_2, multiply each instantaneous value of the complex wave by the sine of twice the corresponding angle of displacement, and take twice their average.

In a similar manner the amplitude of any harmonic, say the n^{th} cosine harmonic, is determined by the formula

$$B_n = \frac{1}{\pi}\int_0^{2\pi} y \cos n\theta\, d\theta, \tag{13}$$

from which we see that B_n is twice the average of $y \cos n\theta$ over one complete cycle.

As a rule, it is advisable to combine the sine and cosine harmonic components of the same frequency, so that the expression for the complex wave shall be either in sine or cosine terms only. This may be accomplished by an application of the well known relationship existing between the sides and angles of a right triangle.

Consider any harmonic component consisting of a sine term and a cosine term, of amplitudes A and B, respectively. Then A and B

may be thought of as the two sides of a right triangle whose hypotenuse is C, as shown in Fig. 1. It will be noted by an inspection of this figure that

$$(14) \quad \begin{cases} A = C\cos\alpha = C\sin\beta, \\ B = C\cos\beta = C\sin\alpha. \end{cases}$$

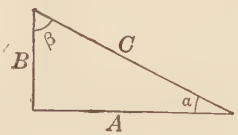

FIG. 1

Hence the harmonic considered may be written either in the form

$$(15) \quad A\sin\theta + B\cos\theta = C\cos\alpha\sin\theta + C\sin\alpha\cos\theta = C\sin(\theta+\alpha),$$

or in the form

$$(16) \quad A\sin\theta + B\cos\theta = C\cos\beta\cos\theta + C\sin\beta\sin\theta = C\cos(\theta-\beta),$$

where

$$(17) \quad C = \sqrt{A^2+B^2}, \quad \alpha = \tan^{-1}\frac{B}{A}, \quad \beta = \tan^{-1}\frac{A}{B}.$$

The sign of the resultant amplitude C, as well as the quadrant in which α and β lie, must be determined with great care from the signs of the amplitudes of the sine and cosine components.

The component harmonics of a complex wave may be thought of as acting independently of one another, *i.e.*, each voltage and current harmonic will produce, independently of the other harmonics, effective dissipative and reactive power, as the case may be. Since the total dissipative and reactive power due the complex waves of voltage and current are, respectively, the sums of the dissipative and reactive powers, produced by their component harmonics, it follows that each voltage and current harmonic acting on a circuit is subject to all the laws governing the relationship between pure sine electromotive forces and currents. The various methods of investigation of pure sine waves are therefore applicable to the investigation of complex ones through their component harmonics.

3. The Complex Number and its Relation to the Harmonic Function. — The instantaneous value of a sine function may be defined in terms of a rotating vector (equal in length to the amplitude of the curve) and its instantaneous position with reference to the positive horizontal as a base line. Both the length of the vector and its position may be specified algebraically with respect to two axes at right angles to each other by the use of complex numbers. Hence, in certain mathematical analysis of alternating

current phenomena, it is possible to avoid the use of the trigonometric functions and employ, instead, simple algebraic expressions involving the imaginary $j = \sqrt{-1}$, whose physical meaning is a rotation of a vector through 90° counter clockwise.

Thus any instantaneous value of a sine curve produced by the projection of a line A, Fig. 2, when A is displaced from the positive horizontal by θ is $A \sin \theta$. The components a and jb along the horizontal and vertical axes, respectively, when the vector A is in this position, completely define not only the numerical value of the vector but its position as well, for

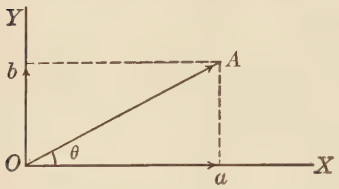

Fig. 2

$$A = \sqrt{a^2 + b^2}, \tag{18}$$

and

$$\theta = \tan^{-1} \frac{b}{a}. \tag{19}$$

Hence A is the vector sum of its components along the two axes, and it may be written in the form

$$A = a + jb, \tag{20}$$

where

$$a = A \cos \theta, \tag{21}$$

and

$$b = A \sin \theta. \tag{22}$$

In other words, the vector is fully identified in length and position by the equation

$$A = (a^2 + b^2)^{\frac{1}{2}} (\cos \theta + j \sin \theta). \tag{23}$$

The usefulness of this representation of a vector by a complex number becomes very apparent when sinusoidal alternating quantities of the same frequency but differing in phase are to be added or subtracted.

Thus, consider two circuits that differ in electric properties, such that with a sinusoidal electromotive force of a definite frequency, impressed upon the two circuits connected in parallel, the current in one branch lags the voltage by an angle ϕ_1, and the current in the other branch leads the voltage by an angle ϕ_2. With reference to the voltage vector as a base line, the relationship between the maximum current values in the two branches is as shown in Fig. 3. Using the complex notation, we may express these currents in magni-

tude and direction, in terms of their components along the horizontal and vertical axes, by the equations

$$I_1 = I_1 \cos \phi_1 - jI_1 \sin \phi_1,$$
$$I_2 = I_2 \cos \phi_2 + jI_2 \sin \phi_2.$$

Applying the ordinary rules of addition, *i.e.*, adding reals and imaginaries, respectively, we get for the total current I_t in the two branches in parallel, the complex expression

$$I_t = (I_1 \cos \phi_1 + I_2 \cos \phi_2) + j(I_2 \sin \phi_2 - I_1 \sin \phi_2).$$

Hence the numerical value of the total current is

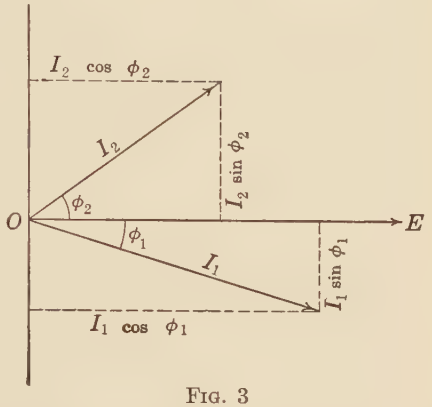

FIG. 3

$$I_t = \sqrt{(I_1 \cos \phi_1 + I_2 \cos \phi_2)^2 + (I_2 \sin \phi_2 - I_1 \sin \phi_1)^2},$$

and its phase with respect to the voltage is

$$\phi_t = \tan^{-1} \frac{I_2 \sin \phi_2 - I_1 \sin \phi_1}{I_1 \cos \phi_1 + I_2 \cos \phi_2}.$$

The total effective current in the circuit is $I_t/\sqrt{2}$. As stated above, the effective current in each branch could have been used conventionally as a vector, and the process of addition carried out as above.

4. Exponential Equivalents of Trigonometric Functions. — The series equivalents of the sine and cosine functions of θ are respectively:

$$(24) \qquad \sin \theta = \frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots,$$

$$(25) \qquad \cos \theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots.$$

Similarly, the series of the exponential $\epsilon^{\pm j\theta}$ is

$$(26) \qquad \epsilon^{\pm j\theta} = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots \pm j\left(\frac{\theta}{1!} - \frac{\theta^5}{3!} + \frac{\theta^7}{5!} - \cdots\right)$$

Comparing this last expression with the series equivalents of the sine and cosine functions, we observe that

$$\epsilon^{j\theta} = \cos\theta + j\sin\theta, \tag{27}$$

and

$$\epsilon^{-j\theta} = \cos\theta - j\sin\theta. \tag{28}$$

Adding equations (27) and (28), and solving for $\cos\theta$, we obtain

$$\cos\theta = \frac{\epsilon^{j\theta} + \epsilon^{-j\theta}}{2}. \tag{29}$$

Subtracting (28) from (27), and solving for $\sin\theta$, we get

$$\sin\theta = \frac{\epsilon^{j\theta} - \epsilon^{-j\theta}}{2j}. \tag{30}$$

These last two expressions are called the *exponential equivalents* of the cosine and sine functions, respectively.

Since any vector is fully identified in magnitude and position with reference to the positive horizontal by writing it in the form $A(\cos\theta \pm j\sin\theta)$, it follows from equations (27) and (28) that the exponential expression of a vector* is $A\,\epsilon^{\pm j\theta}$, which is interpreted as a vector of absolute magnitude A, displaced from the horizontal axis of reference by an angle θ. The numerical value of the vector is called its ***modulus*** and the angle which indicates its position is called its ***argument***. This exponential expression of a vector is frequently written in the form $A\ \angle\theta$.

5. Vector Operations. — Of the various formulas in which a vector may be expressed, one must be chosen that is best adapted to the mathematical operation to which it is to be subjected. Thus, if vectors are to be added or subtracted, the complex form, $a \pm jb = A(\cos\theta \pm j\sin\theta)$ is the most convenient. However, the operations of multiplication, division, involution, evolution, differentiation, and integration are most easily carried out if the vector is expressed as an exponential with an imaginary argument.

6. Addition and Subtraction. — The addition and subtraction of vectors expressed exponentially is performed by converting the expressions into the complex form, and carrying out the process algebraically. Thus we may write

* A vector quantity is often designated by a symbol with a dot or a dash over it, or by a heavy-faced type. In this book, however, we shall use ordinary letters to denote vectors. Thus Z may denote a vector whose absolute value is Z and whose angle is θ. This notation will be used generally in deriving formulas, but when we calculate numerical values, we shall insert the form $Z\epsilon^{j\theta}$, in which Z denotes the absolute value.

$$A\epsilon^{j\theta_1} \pm B\epsilon^{-j\theta_2} = A\cos\theta_1 \pm B\cos\theta_2 + j(A\sin\theta_1 \mp B\sin\theta_2)$$
$$= M + jN = \sqrt{M^2+N^2}\,\epsilon^{\pm j\theta_3},$$

where

$$\theta_3 = \tan^{-1}\frac{N}{M}.$$

7. Multiplication and Division. — These operations are most easily carried out with the vectors expressed exponentially. Thus

$$\frac{a_1+jb_1}{a_2-jb_2}\times(a_3+jb_3) = \frac{\sqrt{a_1^2+b_1^2}\,\epsilon^{j\theta_1}}{\sqrt{a_2^2+b_2^2}\,\epsilon^{-j\theta_2}}\times\sqrt{a_3^2+b_3^2}\,\epsilon^{j\theta_3}$$
$$= \sqrt{\frac{(a_1^2+b_1^2)\,(a_3^2+b_3^2)}{a_2^2+b_2^2}}\,\epsilon^{j(\theta_1+\theta_2+\theta_3)}$$

which, if desired or necessary, may be reconverted into the complex form.

8. Involution and Evolution. — The raising of a vector expression to any power, and the taking of any root of a vector, may readily be done by operating on the vector expressed exponentially. Thus

$$\frac{(a_1+jb_1)^n}{\sqrt[m]{a_2+jb_2}} = \frac{[\sqrt{a_1^2+b_1^2}\,\epsilon^{j\theta_1}]^n}{[\sqrt{a_2^2+b_2^2}\,\epsilon^{j\theta_2}]^{1/m}} = \frac{A^n}{\sqrt[m]{B}}\,\epsilon^{j(n\theta_1-\theta_2/m)},$$

where

$$A = \sqrt{a_1^2+b_1^2}, \qquad B = \sqrt{a_2^2+b_2^2},$$

and

$$\theta_1 = \tan^{-1}\frac{b_1}{a_1}, \qquad \theta_2 = \tan^{-1}\frac{b_2}{a_2}.$$

9. Differentiation and Integration of Vector Expressions. — These operations may be carried out most easily when the vector is expressed either in the trigonometric or exponential form. Thus

$$\frac{d}{dt}(A\epsilon^{j\omega t}) = j\omega A\epsilon^{j\omega t} = \omega A\epsilon^{j(\omega t+\pi/2)},*$$

or

$$\frac{d}{dt}A(\cos\omega t + j\sin\omega t) = \omega A(-\sin\omega t + j\cos\omega t).$$

Similarly the integral of a vector expression is given by

$$\int A\epsilon^{j\omega t}dt = \frac{A}{j\omega}\epsilon^{j\omega t} = -j\frac{A}{\omega}\epsilon^{j\omega t} = \frac{A}{\omega}\epsilon^{j(\omega t-\pi/2)},$$

or

$$\int A(\cos\omega t + j\sin\omega t)\,dt = \frac{A}{\omega}(\sin\omega t - j\cos\omega t).$$

* This follows from the relation $j = 0 + j\,1 = \epsilon^{j\pi/2}$.

10. Trigonometric Relations. — The following table of trigonometric relations will be found useful in obtaining numerical values of functions of angles larger than 90°.

	$-\theta$	$90° \pm \theta$	$180° \pm \theta$	$270° \pm \theta$	$360° \pm \theta$
sin	$-\sin\theta$	$+\cos\theta$	$\mp\sin\theta$	$-\cos\theta$	$\pm\sin\theta$
cos	$+\cos\theta$	$\mp\sin\theta$	$-\cos\theta$	$\pm\sin\theta$	$+\cos\theta$
tan	$-\tan\theta$	$\mp\operatorname{ctn}\theta$	$\pm\tan\theta$	$\mp\operatorname{ctn}\theta$	$\pm\tan\theta$

The following tabulation of formulas will be found useful whenever trigonometric functions are to be converted into one another.

1. $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.
2. $\sin^2\theta + \cos^2\theta = 1$.
3. $\sin\dfrac{\theta}{2} = \sqrt{\dfrac{1-\cos\theta}{2}}$.
4. $\cos\dfrac{\theta}{2} = \sqrt{\dfrac{1+\cos\theta}{2}}$.
5. $\tan\dfrac{\theta}{2} = \sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}}$.
6. $\sin 2\theta = 2\sin\theta\cos\theta$.
7. $\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \cos^2\theta - \sin^2\theta$.
8. $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$.
9. $\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi$.
10. $\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$.
11. $\tan(\theta \pm \phi) = \dfrac{\tan\theta \pm \tan\phi}{1 \mp \tan\theta\tan\phi}$.
12. $j\sin\theta = \dfrac{\epsilon^{j\theta} - \epsilon^{-j\theta}}{2}$.
13. $\cos\theta = \dfrac{\epsilon^{j\theta} + \epsilon^{-j\theta}}{2}$.
14. $j\tan\theta = \dfrac{\epsilon^{j\theta} - \epsilon^{-j\theta}}{\epsilon^{j\theta} + \epsilon^{-j\theta}}$.

CHAPTER III

HYPERBOLIC FUNCTIONS

Under well-defined conditions, there exist relations between electrical quantities that are best expressed in terms of the hyperbolic or their equivalent exponential functions. Examples of this are the rise or decay of current in an inductive circuit with constant unidirectionally acting electromotive force impressed upon it or removed from it, respectively; the decay of the transient current or the growth of quantity of electricity in a dielectric circuit with unidirectional electromotive force impressed upon it; the fall of, or the attenuation of, potential and current values along a transmission line. A familiarity with these functions, their properties, relations between them and with the circular trigonometric functions, is important, inasmuch as it enables us to investigate and interpret problems dealing with conditions similar to those mentioned above.

11. Analogy between the Hyperbolic and Circular Trigonometric Functions. — The names ***hyperbolic sine, hyperbolic cosine, hyperbolic tangent,*** etc., have been given to certain ratios between quantities that exist with reference to the rectangular hyperbola, by analogy with similar ratios that exist between quantities with reference to the circle.

Consider a circular sector OPQ, Fig. 4, of radius r. The area A_c of this circular sector is

$$(1) \qquad A_c = \frac{r}{2} \text{ arc } PQ;$$

and since the length of the arc is

$$\text{arc } PQ = \theta r,$$

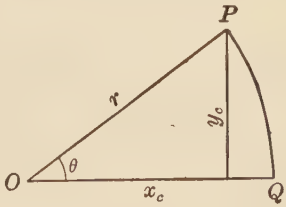

FIG. 4

where θ is expressed in radians, it follows that

$$(2) \qquad A_c = \frac{1}{2}\theta r^2,$$

whence

$$(3) \qquad \theta = \frac{2A_c}{r^2}.$$

Referring to Fig. 4, we see that

$$\sin\theta = \frac{y_c}{r}, \quad \text{and} \quad \cos\theta = \frac{x_c}{r}.$$

Substituting for θ its value from (3), we get

$$\sin\frac{2A_c}{r^2} = \frac{y_c}{r}, \tag{4}$$

and

$$\cos\frac{2A_c}{r^2} = \frac{x_c}{r}, \tag{5}$$

whence

$$\cos\frac{2A_c}{r^2} + j\sin\frac{2A_c}{r^2} = \frac{x_c}{r} + j\frac{y_c}{r}, \tag{6}$$

which by equation (27), Chapter II, may be written

$$\epsilon^{j2A_c/r^2} = \frac{x_c}{r} + j\frac{y_c}{r}. \tag{7}$$

An analogous expression may be derived for the rectangular hyperbola as follows. Consider the sector OMN of the rectangular hyperbola MNP, shown in Fig. 5. Obviously the area of this sector is equal to the area of the right triangle OMQ, less the area under the hyperbolic curve. Letting $MQ = y_h$ and $OQ = x_h$, we have

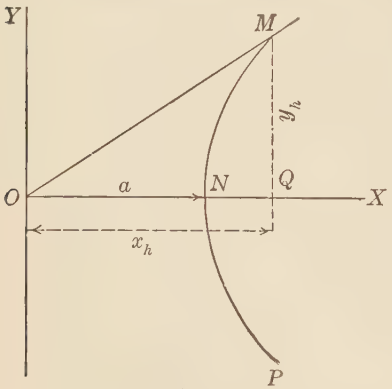

Fig. 5

$$\text{area } OMN = \frac{x_h y_h}{2} - \text{area } MNQ. \tag{8}$$

The equation of the rectangular hyperbola with reference to the Cartesian system of coordinates, is

$$y = \sqrt{x^2 - a^2} \tag{9}$$

where a, the semi-major axis $= ON$. Hence

$$\text{area } MNQ = \int_a^x y\,dx = \int_a^x \sqrt{x^2 - a^2}\,dx$$

$$= \frac{1}{2}\left\{ x\sqrt{x^2 - a^2} - a^2 \log_\epsilon (x + \sqrt{x^2 - a^2}) \right\}_a^x$$

and substituting limits, we get

$$\text{area } MNQ = \frac{1}{2}\left[x\sqrt{x^2 - a^2} - a^2 \log_\epsilon (x + \sqrt{x^2 - a^2})\right] + \frac{1}{2} a^2 \log_\epsilon a$$

$$= \frac{1}{2}\left\{ x\sqrt{x^2 - a^2} - a^2 \log_\epsilon \frac{x + \sqrt{x^2 - a^2}}{a} \right\}$$

$$= \frac{1}{2}\left\{ x_h y_h - a^2 \log_\epsilon \frac{x_h + y_h}{a} \right\} \tag{10}$$

Substituting this result in equation (8), we have

$$\text{area } OMN = A_h = \frac{a^2}{2} \log_\epsilon \left(\frac{x_h}{a} + \frac{y_h}{a}\right), \tag{11}$$

whence

$$\log_\epsilon \left(\frac{x_h}{a} + \frac{y_h}{a}\right) = \frac{2\,A_h}{a^2}$$

or

$$\epsilon^{2\,A_h/a^2} = \frac{x_h}{a} + \frac{y_h}{a}. \tag{12}$$

This expression is analogous to equation (7) obtained for the circular sector. Extending the analogy further x_h/a is called the ***hyperbolic cosine***, and is denoted by the symbol cosh and y_h/a is called ***hyperbolic sine***, and is denoted by the symbol sinh. Hence, equation (12) may be written, by analogy with equation (7)

$$\epsilon^{2\,A_h/a^2} = \cosh \frac{2\,A_h}{a^2} + \sinh \frac{2\,A_h}{a^2} = \epsilon^u, \tag{13}$$

where the quantity

$$u = \frac{2\,A_h}{a^2} \tag{14}$$

may be thought of as a ***hyperbolic angle***, measured (by analogy to the circular angle) in ***hyperbolic radians***.

From equations (12), (13), and (14), it follows that

$$\cosh u = \frac{x_h}{a}, \quad \text{and} \quad \sinh u = \frac{y_h}{a} \tag{15}$$

Since by equation (3)

$$\theta = \frac{2\,A_c}{r^2},$$

we have $\theta = 2$ when $A_c = 1$ and $r = 1$, it follows that the *radian*, as a unit of circular measure, may be defined as the *arc of a circle*

described by a unit radius sweeping over one-half unit area. By analogy, equation (14)

$$u = \frac{2\,A_h}{a^2}$$

may be used to define the *hyperbolic radian, as the hyperbolic angle that obtains with a hyperbolic sector of one-half unit area and of semi-major axis equal to unity.* In other words, *a hyperbolic angle in hyperbolic radians is numerically equal to twice the area of a hyperbolic sector of unit semi-major axis.*

12. Relations between Hyperbolic Functions. — With the fundamental analogies between the circular and hyperbolic functions outlined above, it is now possible to develop a series of relations between hyperbolic functions, similar to the relations that exist between circular trigonometric functions. Thus, from the general equation of a rectangular hyperbola, equation (9), which is generally written in the form

$$\frac{x^2}{a} - \frac{y^2}{a} = 1, \tag{15a}$$

and from equations (15), we may obtain the relation

$$\cosh^2 u - \sinh^2 u = 1.$$

Also, by analogy with the circular trigonometric tangent, we write

$$\tanh u = \frac{\sinh u}{\cosh u},$$

and

$$\operatorname{ctnh} u = \frac{1}{\tanh u}.$$

The following brief table will be found useful in evaluating numerical values of negative hyperbolic angles, and in converting hyperbolic functions from one form into another.

$$\sinh(-u) = -\sinh u. \tag{15b}$$

$$\cosh(-u) = +\cosh u. \tag{15c}$$

$$\tanh(-u) = -\tanh u. \tag{15d}$$

$$\tanh u = \frac{\sinh h}{\cosh u}. \tag{15e}$$

$$\cosh^2 u - \sinh^2 u = 1. \tag{15f}$$

$$\cosh^2 u = \frac{1}{1 - \tanh^2 u}. \tag{15g}$$

(15h) $\sinh^2 u = \dfrac{1}{\operatorname{ctnh}^2 u - 1}.$

(15i) $\sinh 2u = 2 \sinh u \cosh u.$

(15j) $\cosh 2u = \cosh^2 u + \sinh^2 u = 1 + 2\sinh^2 u = 2\cosh^2 u - 1.$

(15k) $\tanh 2u = \dfrac{2 \tanh u}{1 + \tanh^2 u}.$

(15l) $\sinh \dfrac{u}{2} = \sqrt{\dfrac{\cosh u + 1}{2}}.$

(15m) $\cosh \dfrac{u}{2} = \sqrt{\dfrac{\cosh u - 1}{2}}.$

(15n) $\tanh \dfrac{u}{2} = \sqrt{\dfrac{\cosh u - 1}{\cosh u + 1}}.$

(15o) $\sinh (u \pm v) = \sinh u \cosh v \pm \cosh u \sinh v.$

(15p) $\cosh (u \pm v) = \cosh u \cosh v \pm \sinh u \sinh v.$

(15q) $\tanh (u \pm v) = \dfrac{\tanh u \pm \tanh v}{1 \pm \tanh u \tanh v}.$

(15r) $\dfrac{d}{du} \sinh u = \cosh u.$

(15s) $\dfrac{d}{du} \cosh u = \sinh u.$

(15t) $\dfrac{d}{du} \tanh u = \dfrac{1}{\cosh^2 u}.$

13. The Exponential Equivalent of the Hyperbolic Function. — By equations (13) and (15) the sum and the difference of the hyperbolic cosine and sine may be written in the forms

(16) $$\epsilon^u = \cosh u + \sinh u,$$

(17) $$\epsilon^{-u} = \cosh u - \sinh u.$$

Adding these two expressions, and solving for $\cosh u$, we get

(18) $$\cosh u = \frac{\epsilon^u + \epsilon^{-u}}{2}.$$

Subtracting (17) from (16) and solving for $\sinh u$, we find

(19) $$\sinh u = \frac{\epsilon^u - \epsilon^{-u}}{2},$$

whence

(20) $$\tanh u = \frac{\epsilon^u - \epsilon^{-u}}{\epsilon^u + \epsilon^{-u}}.$$

14. Relationship between Circular Trigonometric and Hyperbolic Functions.[1] — By means of the exponential equivalents it is possible to establish definite relations between hyperbolic or circular angles ju or $j\theta$. Such imaginary quantities occur very frequently in problems related to long distance electric transmission of energy.

Thus, if we substitute $j\theta$ for u, in equation (18), we find

$$\cosh j\theta = \frac{\epsilon^{j\theta} + \epsilon^{-j\theta}}{2} = \cos\theta, \tag{21}$$

which shows that the hyperbolic cosine of an imaginary hyperbolic angle is equivalent to the circular trigonometric cosine of a real circular angle equal numerically to the imaginary hyperbolical one.

Similarly, if we substitute $j\theta$ for u in (19), we obtain

$$\sinh j\theta = \frac{\epsilon^{j\theta} - \epsilon^{-j\theta}}{2} = j\sin\theta. \tag{22}$$

Adding equations (21) and (22), we find

$$\cosh j\theta + \sinh j\theta = \cos\theta + j\sin\theta,$$

which means that the sum of the hyperbolic cosine and sine of an imaginary hyperbolic angle is a vector quantity of unit length. displaced from the positive horizontal as a reference line by an angle equal numerically to the imaginary hyperbolic angle.

It also follows from equations (21) and (22) that

$$\tanh j\theta = \frac{\sinh j\theta}{\cosh j\theta} = j\tanh\theta. \tag{23}$$

In a similar manner, the circular trigonometric functions of pure imaginaries may be evaluated by converting them into hyperbolic functions of real angles. Thus if in the expression

$$\sin\theta = \frac{\epsilon^{j\theta} - \epsilon^{-j\theta}}{2j},$$

we substitute ju for θ, since $j^2 = -1$, we find

$$\sin ju = \frac{\epsilon^{j^2u} - \epsilon^{-j^2u}}{2j} = j\frac{\epsilon^{u} - \epsilon^{-u}}{2} = j\sinh u. \tag{24}$$

Making the same substitution in the expression for the cosine, we get

[1] An interesting paper entitled *Physical Interpretation of Complex Angles and their Functions*, by Aram Boyajian, will be found in the JOURNAL OF A. I. E. E., vol. XLII, No. 2, Feb., 1923.

$$\cos ju = \frac{\epsilon^{j^2u} + \epsilon^{-j^2u}}{2} = \frac{\epsilon^{-u} + \epsilon^{u}}{2} = \cosh u, \tag{25}$$

and from equations (24) and (25), it follows that

$$\tan ju = j \tanh u. \tag{26}$$

15. Trigonometric and Hyperbolic Functions of Complex Quantities. — Quantities frequently occur in physical problems which are growing or decaying hyperbolically while they vary harmonically. Such quantities are subject to trigonometric and hyperbolic laws of variation simultaneously. Electrical transmission of energy is probably the most important of all such problems. The resulting mathematical expressions involve either hyperbolic or trigonometric functions of complex numbers. The numerical evaluation and the physical interpretation of such expressions may be obtained through the relations between hyperbolic and trigonometric functions of imaginaries developed in the preceding section.

While complex angles of the form $\theta \pm ju$ or $u \pm j\theta$, where θ and u are circular trigonometric and hyperbolic angles, respectively, may not have any physical meaning, the trigonometric and hyperbolic functions of such complex expressions may not only be evaluated numerically, but have also a definite physical meaning, as is shown below.

Expanding the sine function of a complex angle, we get

$$\sin (\theta \pm ju) = \sin \theta \cos ju \pm \cos \theta \sin ju$$

and since by (24) and (25), respectively,

$$\sin ju = j \sinh u, \quad \text{and} \quad \cos ju = \cosh u,$$

we may write

$$\sin (\theta \pm ju) = \sin \theta \cosh u \pm j \cos \theta \sinh u. \tag{27}$$

This shows that the trigonometric sine of a complex angle is equivalent numerically to a complex number of the form $(a \pm jb)$. In other words, it is a vector of definite magnitude and definite position with reference to a base line. It follows that equation (27) may be written in the form

$$\sin (\theta \pm ju) = \sqrt{\sin^2 \theta \cosh^2 u + \cos^2 \theta \sinh^2 u}\ \epsilon^{\pm j\phi_1},$$

where

$$\phi_1 = \tan^{-1} \frac{\cos \theta \sinh u}{\sin \theta \cosh u} = \tan^{-1} \frac{\tanh u}{\tan \theta}. \tag{28}$$

Moreover, since

$$\sin^2\theta = 1 - \cos^2\theta,$$

and

$$\sinh^2 u = \cosh^2 u - 1,$$

it follows by substituting in the preceding equation that

$$\sin(\theta \pm ju) = \sqrt{(1 - \cos^2\theta)\cosh^2 u + \cos^2\theta\,(\cosh^2 u - 1)}\;\epsilon^{\pm j\phi_1}$$

which reduces to the simpler form

$$\sin(\theta \pm ju) = \sqrt{\cosh^2 u - \cos^2\theta}\;\epsilon^{\pm j\phi_1}. \tag{29}$$

Similarly, the trigonometric cosine of a complex angle may be evaluated as follows:

$$\begin{aligned}\cos(\theta \pm ju) &= \cos\theta\cos ju \mp \sin\theta\sin ju\\ &= \cos\theta\cosh u \mp j\sin\theta\sinh u\\ &= \sqrt{\cos^2\theta\cosh^2 u + \sin^2\theta\sinh^2 u}\;\epsilon^{\mp j\phi_2}\\ &= \sqrt{(1-\sin^2\theta)\cosh^2 u + \sin^2\theta\,(\cosh^2 u - 1)}\;\epsilon^{\mp j\phi_2}\\ &= \sqrt{\cosh^2 u - \sin^2\theta}\;\epsilon^{\mp j\phi_2}, \qquad (30)\end{aligned}$$

where

$$\phi_2 = \tan^{-1}\frac{\sin\theta\sinh u}{\cos\theta\cosh u} = \tan^{-1}(\tanh u\tan\theta). \tag{31}$$

From equations (29) and (30), we find

$$\begin{aligned}\tan(\theta \pm ju) &= \frac{\sin(\theta \pm ju)}{\cos(\theta \pm ju)}\\ &= \sqrt{\frac{\cosh^2 u - \cos^2\theta}{\cosh^2 u - \sin^2\theta}}\;\epsilon^{j(\pm\phi_1 \pm \phi_2)} \qquad (32)\end{aligned}$$

We may conclude from the above that the circular trigonometric functions of complex angles are vector quantities of definite magnitude and definite position with reference to a base line.

In a similar manner, the hyperbolic functions of complex angles can be evaluated numerically and interpreted physically. Expanding the hyperbolic sine of the complex angle $u \pm j\theta$ we get

$$\sinh(u \pm j\theta) = \sinh u\cosh j\theta \pm \cosh u\sinh j\theta.$$

By equations (21) and (22) we have

$$\cosh j\theta = \cos\theta, \quad \text{and} \quad \sinh j\theta = j\sin\theta.$$

Substituting in the above equation, we find

$$\sinh(u \pm j\theta) = \sinh u\cos\theta \pm j\cosh u\sin\theta.$$

This is a complex quantity of the form $a \pm jb$, which indicates that the hyperbolic sine of a complex angle is a vector whose modulus is

$$\sqrt{\sinh^2 u \cos^2 \theta + \cosh^2 u \sin^2 \theta}$$

and whose argument is

$$\beta_2 = \tan^{-1} \frac{\cosh u \sin \theta}{\sinh u \cos \theta} = \tan^{-1} \frac{\tan \theta}{\tanh u}. \tag{33}$$

Moreover, since

$$\sinh^2 u = \cosh^2 u - 1,$$

and

$$\sin^2 \theta = 1 - \cos^2 \theta,$$

it follows that

$$\sinh (u \pm j\theta) = \sqrt{(\cosh^2 u - 1) \cos^2 \theta + (1 - \cos^2 \theta) \cosh^2 u}\ \epsilon^{\pm j\beta_2},$$

which reduces to the simpler form

$$\sinh (u \pm j\theta) = \sqrt{\cosh^2 u - \cos^2 \theta}\ \epsilon^{\pm j\beta_2}. \tag{34}$$

The hyperbolic cosine of a complex angle may be obtained in a similar manner, as follows:

$$\begin{aligned} \cosh (u \pm j\theta) &= \cosh u \cosh j\theta \pm \sinh u \sinh j\theta \\ &= \cosh u \cos \theta \pm j \sinh u \sin \theta. \\ &= \sqrt{\cosh^2 u - \sin^2 \theta}\ \epsilon^{\pm j\beta_1}, \end{aligned} \tag{35}$$

where

$$\beta_1 = \tan^{-1} \frac{\sinh u \sin \theta}{\cosh u \cos \theta} = \tan^{-1} (\tan \theta \tanh u). \tag{36}$$

From equations (34) and (35), it follows that

$$\begin{aligned} \tanh (u \pm j\theta) &= \frac{\sinh (u \pm j\theta)}{\cosh (u \pm j\theta)} \\ &= \sqrt{\frac{\cosh^2 u - \cos^2 \theta}{\cosh^2 u - \sin^2 \theta}}\ \epsilon^{j(\pm \beta_2 \mp \beta_1)} \end{aligned} \tag{37}$$

16. Exponential Equivalents of Circular Trigonometric and Hyperbolic Functions of Complex Numbers. — It is frequently necessary for analytical purposes to convert trigonometric and hyperbolic functions of complex quantities into exponential form. This operation may be carried out easily from the relations be-

tween hyperbolic and trigonometric functions of real numbers and their respective exponential equivalents. Thus we have

$$(38)\quad \begin{cases} j\sin(\theta \pm ju) = \dfrac{\epsilon^{j(\theta \pm ju)} - \epsilon^{-j(\theta \pm ju)}}{2} \\ \qquad\qquad = \dfrac{\epsilon^{\mp u + j\theta} - \epsilon^{\pm u - j\theta}}{2}, \end{cases}$$

$$(39)\quad \begin{cases} \cos(\theta \pm ju) = \dfrac{\epsilon^{j(\theta \pm ju)} + \epsilon^{-j(\theta \pm ju)}}{2} \\ \qquad\qquad = \dfrac{\epsilon^{\mp u + j\theta} + \epsilon^{\pm u - j\theta}}{2}, \end{cases}$$

and from equations (38) and (39) we find

$$(40)\quad j\tan(\theta \pm ju) = \frac{\epsilon^{\mp u + j\theta} - \epsilon^{\pm u - j\theta}}{\epsilon^{\mp u + j\theta} + \epsilon^{\pm u - j\theta}}.$$

Similarly, for the hyperbolic functions, we have

$$(41)\quad \sinh(u \pm j\theta) = \frac{\epsilon^{u \pm j\theta} - \epsilon^{-(u \pm j\theta)}}{2},$$

$$(42)\quad \cosh(u \pm ju) = \frac{\epsilon^{u \pm j\theta} + \epsilon^{-(u \pm j\theta)}}{2},$$

and, from (41) and (42), we find

$$(43)\quad \tanh(u \pm j\theta) = \frac{\epsilon^{u \pm j\theta} - \epsilon^{-(u \pm j\theta)}}{\epsilon^{u \pm j\theta} + \epsilon^{-(u \pm j\theta)}}.$$

CHAPTER IV

GENERAL EQUATIONS OF ELECTRIC WAVE PROPAGATION

In the mathematical analysis given in this chapter, the assumption is made that the electromotive force impressed at the sending end of a transmission line is sinusoidal in character, and that the transients incident on starting have died out, so that conditions have become steady.

17. Fundamental Relations. — Consider a transmission line of length S miles, with uniformly distributed resistance, inductance, electrostatic capacity, and leakage conductance. As shown in Fig. 6, let

dS = element of line conductor,
r = resistance per mile of conductor,
l = inductance per mile of conductor;

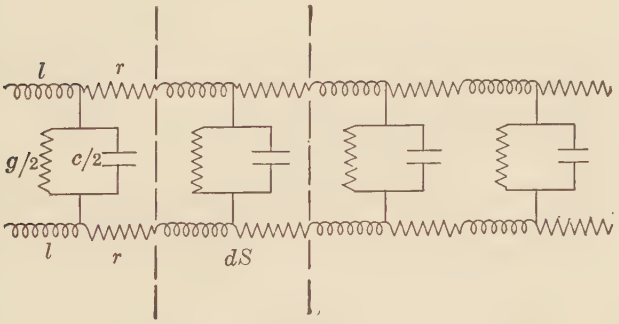

FIG. 6

then

$r\,dS$ = resistance of line conductor of length dS,

and

$l\,dS$ = inductance of line conductor of length dS.

If

i = current in conductor element dS,

it follows that the change in potential due to resistance and inductance is

$$de = ir\,dS + ldS\,\frac{di}{dt}, \tag{1}$$

whence the rate of change of potential with respect to distance along the line is

$$\frac{de}{dS} = ir + l\frac{di}{dt}. \tag{2}$$

Let

g = effective leakage conductance (including equivalent effect of dielectric hysteresis) per mile of line conductor,

and

c = capacitance per mile of line conductor,

we have

$g\,dS$ = leakage conductance of conductor element dS,

and

$c\,dS$ = capacitance of conductor element dS.

If

dq = electric charge on conductor element dS,

it follows that

$$dq = c\,dS\,de;$$

hence the charging current due to the capacitance of conductor element dS is

$$\frac{dq}{dt} = c\,dS\frac{de}{dt}.$$

Since e is the voltage between conductors at the conductor element dS, it follows that the leakage current is $g\,e\,dS$, and that the change in current due to both leakage and capacitance is

$$di = ge\,dS + cdS\frac{de}{dt}; \tag{3}$$

whence the rate of change of current with respect to distance along the line is

$$\frac{di}{dS} = ge + c\frac{de}{dt}. \tag{4}$$

18. Mathematical Analysis. — Differentiating equations (2) and (4), first with respect to distance S and then with respect to time t, we obtain

$$\frac{d^2e}{dS^2} = r\frac{di}{dS} + l\frac{d^2i}{dS\,dt}, \tag{5}$$

$$\frac{d^2i}{dS^2} = g\frac{de}{dS} + c\frac{d^2e}{dS\,dt}, \tag{6}$$

$$\frac{d^2e}{dS\,dt} = r\frac{di}{dt} + l\frac{d^2i}{dt^2}, \tag{7}$$

$$\frac{d^2i}{dS\,dt} = g\frac{de}{dt} + c\frac{d^2e}{dt^2}, \tag{8}$$

Substituting in equation (5), the value of $d^2i/dS\,dt$ as given in equation (8), and the value of di/dS from equation (4), we get

$$\frac{d^2e}{dS^2} = rge + rc\frac{de}{dt} + lg\frac{de}{dt} + lc\frac{d^2e}{dt^2}.$$

Collecting terms, we find

$$\frac{d^2e}{dS^2} = lc\frac{d^2e}{dt^2} + (rc + lg)\frac{de}{dt} + rge. \tag{9}$$

Similarly, substituting in equation (6), the value of $d^2e/dS\,dt$ from equation (7), and the value of de/dS as given in equation (2), and collecting terms, we get

$$\frac{d^2i}{dS^2} = lc\frac{d^2i}{dt^2} + (rc + lg)\frac{di}{dt} + rgi. \tag{10}$$

Equations (9) and (10) are differential equations of the second order, with respect to S and t.

Since both the voltage and the current at the sending end of the line are assumed to be sine functions of the time, it is obvious that they will be sine functions of the time at any point along the line. Since the instantaneous values may be expressed in terms of maximum values and phase angle, equations (9) and (10) may be written, respectively, in the forms

$$\frac{d^2e}{dS^2} = lc\frac{d^2}{dt^2}E\epsilon^{j\omega t} + (rc + lg)\frac{d}{dt}E\epsilon^{j\omega t} + rg\,E\,\epsilon^{j\omega t}, \tag{11}$$

$$\frac{d^2i}{dS^2} = lc\frac{d^2}{dt^2}I\epsilon^{j(\omega t \pm \theta)} + (rc + lg)\frac{d}{dt}I\epsilon^{j(\omega t \pm \theta)} + rgI\epsilon^{j(\omega t \pm \theta)}, \tag{12}$$

where E and I are maximum values of voltage and current, respectively, where $\omega = 2\pi f$, f being the frequency of the supply, and where θ is the time phase angle between voltage and current at any point along the line. But we have

$$\frac{d}{dt}E\epsilon^{j\omega t} = j\omega E\epsilon^{j\omega t}, \tag{13}$$

and

$$\frac{d^2}{dt^2}E\epsilon^{j\omega t} = j^2\omega^2 E\epsilon^{j\omega t} = -\omega^2 E\epsilon^{j\omega t} \tag{14}$$

Moreover,

$$\frac{d}{dt} I\epsilon^{j(\omega t \pm \theta)} = j\omega I\epsilon^{j(\omega t \pm \theta)}, \tag{15}$$

and

$$\frac{d^2}{dt^2} I\epsilon^{j(\omega t \pm \theta)} = j^2\omega^2 I\epsilon^{j(\omega t \pm \theta)} = -\omega^2 I\epsilon^{j(\omega t \pm \theta)}. \tag{16}$$

Substituting these expressions in equation (11), we obtain

$$\frac{d^2e}{dS^2} = [-\omega^2 lc\, E + (rc + lg)\, j\omega E + rgE]\, \epsilon^{j\omega t}.$$

Rearranging we may write this equation in the form

$$\frac{d^2e}{dS^2} = [(r + jl\omega)\ (g + jc\omega)]\, E\epsilon^{j\omega t}. \tag{17}$$

Performing the same operations on equation (12), we get

$$\frac{d^2i}{dS^2} = [(r + jl\omega)\ (g + jc\omega)]\, I\epsilon^{j(\omega t \pm \theta)}. \tag{18}$$

It will be more convenient to use the physical constant of the line per mile of line, instead of per mile of conductor. Accordingly, if

R = resistance per mile of line (loop mile or two miles of conductor)

L = inductance per mile of line,

C = capacitance per mile of line, and

G = leakage conductance per mile of line,

we shall have

$$R = 2\,r,\ L = 2\,l,\ C = \frac{c}{2},\ G = \frac{g}{2}.$$

Substituting in equations (17) and (18), we get

$$\frac{d^2e}{dS^2} = (R + jL\omega)\ (G + jC\omega)\ E\epsilon^{j\omega t}, \tag{19}$$

$$\frac{d^2i}{dS^2} = (R + jL\omega)\ (G + jC\omega)\ I\epsilon^{j(\omega t \pm \theta)}. \tag{20}$$

The quantity

$$R + jL\omega = z$$

is called *linear line impedance*, and the quantity

$$G + jC\omega = y$$

is called *linear line admittance* per mile of line.

Using the above symbols for the linear line admittance and impedance, equations (19) and (20) may be written in the abbreviated form

$$\frac{d^2e}{dS^2} = zyE\epsilon^{j\omega t}, \tag{21}$$

$$\frac{d^2i}{dS^2} = zyI\epsilon^{j(\omega t \pm \theta)}. \tag{22}$$

To find the variation in voltage and current values with distance from the sending end, for any definite instant of time, the above two equations must be integrated, S being the only variable.

The integral solutions of these two equations are

$$e = [A_1 \cosh S\sqrt{zy} + B_1 \sinh S\sqrt{zy}]\,\epsilon^{j\omega t}, \tag{23}$$

$$i = [A_2 \cosh S\sqrt{zy} + B_2 \sinh S\sqrt{zy}]\,\epsilon^{j(\omega t \pm \theta)}, \tag{24}$$

where A_1, B_1, A_2, B_2 are the constants of integration whose values must be determined from the boundary conditions of the problem. Thus, letting

S_s = the distance from the sending end,
E_s = maximum instantaneous voltage at the sending end,
I_s = maximum instantaneous current at the sending end,

we find, when $S_s = 0$,

$$e = E_s\epsilon^{j\omega t}, \qquad \text{and} \qquad i = I_s\epsilon^{j(\omega t \pm \theta)}.$$

Hence, when $S_s = 0$, *i.e.*, at the sending end, we have

$$\cosh S_s\sqrt{zy} = 1, \qquad \sinh S_s\sqrt{zy} = 0.$$

Under these conditions, equation (23) becomes

$$E_s\epsilon^{j\omega t} = A_1\epsilon^{j\omega t},$$

whence

$$A_1 = E_s. \tag{25}$$

Equation (24) becomes also

$$I_s\epsilon^{j(\omega t \pm \theta)} = A_2\epsilon^{j(\omega t \pm \theta)},$$

whence

$$A_2 = I_s. \tag{26}$$

To determine the constants B_1 and B_2, substitute in equation (2) the value of i given in (24), using the negative sign to indicate a decrease in voltage as we proceed from the sending end of the line.

Using also the value of A_2 from (26), we get

$$(27)\quad \frac{de}{dS_s} = -(R + jL\omega)\,[I_s \cosh S_s\sqrt{zy} + B_2 \sinh S_s\sqrt{zy}]\,\epsilon^{j(\omega t \pm \theta)}$$

Similarly, differentiating equation (23) with respect to S_s and using the value of A_1 from (25), we obtain

$$(28)\quad \frac{de}{dS_s} = [E_s\sqrt{zy}\,\sinh S_s\sqrt{zy} + B_1\sqrt{zy}\,\cosh S_s\sqrt{zy}]\epsilon^{j\omega t}.$$

Substituting in (4) the value of e given in (23), using $A_1 = E_s$ by (25), we get

$$(29)\quad \frac{di}{dS_s} = -(G + jC\omega)\,[E_s \cosh S_s\sqrt{zy} + B_1 \sinh S_s\sqrt{zy}]\,\epsilon^{j\omega t}.$$

Differentiating (24) with respect to S_s and using $A_2 = I_s$ by (26), we obtain

$$(30)\quad \frac{di}{dS_s} = [I_s\sqrt{zy}\,\sinh S_s\sqrt{zy} + B_2\sqrt{zy}\,\cosh S_s\sqrt{zy}]\epsilon^{j(\omega t \pm \theta)}.$$

Equating the right members of (27) and (28), and those of (29) and (30), respectively, we get two equations in B_1 and B_2, which may be solved simultaneously. This gives

$$(31)\quad B_2\epsilon^{j(\omega t \pm \theta)} = \frac{-E_s\sqrt{zy}}{R + jL\omega}\,\epsilon^{j\omega t}$$

and

$$(32)\quad B_1\epsilon^{j\omega t} = -I_s\frac{R + jL\omega}{\sqrt{zy}}\,\epsilon^{j(\omega t \pm \theta)}.$$

Substituting the values of B_2 and B_1 as obtained in (31) and (32) in (24) and (23), respectively, we get

$$e = E_s\epsilon^{j\omega t}\cosh S_s\sqrt{zy} - I_s\frac{R + jL\omega}{\sqrt{zy}}\,\epsilon^{j(\omega t \pm \theta)}\sinh S_s\sqrt{zy},$$

$$i = I_s\epsilon^{j(\omega t \pm \theta)}\cosh S_s\sqrt{zy} - E_s\frac{\sqrt{zy}}{R + jL\omega}\,\epsilon^{j\omega t}\sinh S_s\sqrt{zy}.$$

Since $R + jL\omega = z$ and $G + jC\omega = y$, it follows that

$$(33)\quad \frac{R + jL\omega}{\sqrt{zy}} = \sqrt{\frac{R + jL\omega}{G + jC\omega}},$$

which we shall denote by Z_0, and that

$$\sqrt{zy} = \sqrt{(R + jL\omega)\ (G + jC\omega)} \tag{34}$$

which we shall denote by p. Then the last two equations may be written in the abbreviated form

$$e = E_s\epsilon^{j\omega t} \cosh pS_s - I_sZ_0\epsilon^{j(\omega t \pm \theta)} \sinh pS_s, \tag{35}$$

$$i = I_s\epsilon^{j(\omega t \pm \theta)} \cosh pS_s - \frac{E_s}{Z_0}\epsilon^{j\omega t} \sinh pS_s. \tag{36}$$

These two equations are called *general transmission equations,* and they represent the instantaneous values of potential and current along a transmission line at any point distant S_s miles from the sending end.

19. Physical Interpretation of the General Transmission Equations. — The preceding equations for voltage and current indicate that both voltage and current at any point along the line are, respectively, the resultant of two sine functions of time of the same frequency as the supply.

They also indicate that the maximum values of the voltage and current as sine functions of time, are hyperbolic functions of a complex angle, whose value depends upon the constants of the line and the distance from the sending end.

Considering then the variations of only maximum values with distance along the line, and keeping in mind that the quantities involved are vector quantities, equations (35) and (36) may be written, respectively, in the forms

$$E = E_s \cosh pS_s - I_sZ_0 \sinh pS_s, \tag{37}$$

$$I = I_s \cosh pS_s - \frac{E_s}{Z_0} \sinh pS_s. \tag{38}$$

Using the exponential equivalents of the hyperbolic functions (37) may also be written in the form

$$E = E_s \frac{\epsilon^{pS_s} + \epsilon^{-pS_s}}{2} - I_sZ_0 \frac{\epsilon^{pS_s} - \epsilon^{-pS_s}}{2},$$

or

$$E = \frac{E_s - I_sZ_0}{2}\epsilon^{pS_s} + \frac{E_s + I_sZ_0}{2}\epsilon^{-pS_s}. \tag{39}$$

Similarly, the equation (38) may be written in the form

$$I = \frac{I_sZ_0 - E_s}{2\,Z_0}\epsilon^{pS_s} + \frac{I_sZ_0 + E_s}{2\,Z_0}\epsilon^{-pS_s}. \tag{40}$$

20. The Propagation Constant. — The quantity p in the preceding equations is called *propagation constant*. Since we have

$$R + jL\omega = \sqrt{R^2 + L^2\omega^2}\, \epsilon^{j\alpha_1}, \tag{41}$$

where

$$\alpha_1 = \tan^{-1}\frac{L\omega}{R}, \tag{41 a}$$

and

$$G + jC\omega = \sqrt{G^2 + C^2\omega^2}\, \epsilon^{j\alpha_2}, \tag{42}$$

$$\alpha_2 = \tan^{-1}\frac{C\omega}{G}, \tag{42 a}$$

it follows that the propagation constant

$$p = \sqrt{(R + jL\omega)\,(G + jC\omega)}$$

may be expressed in the form

$$p = \sqrt[4]{(R^2 + L^2\omega^2)\,(G^2 + C^2\omega^2)}\, \epsilon^{j\delta}, \tag{43}$$

where

$$\delta = \frac{\alpha_1 + \alpha_2}{2}. \tag{43 a}$$

Equation (43) shows that the propagation constant is a vector quantity and as such may be written in the form

$$p = \sqrt[4]{(R_2 + L^2\omega^2)\,(G^2 + C^2\omega^2)}\,[\cos\delta + j\sin\delta].$$

If we set the real component equal to a:

$$p\cos\delta = a, \tag{44}$$

and the imaginary component equal to v:

$$p\sin\delta = v, \tag{45}$$

the propagation constant may be written in the abbreviated form

$$p = a + jv. \tag{46}$$

Using this expression for p in equations (39) and (40), we obtain

$$E = \left[\frac{E_s - I_sZ_0}{2}\epsilon^{aS_s}\right]\epsilon^{jvS_s} + \left[\frac{E_s + I_sZ_0}{2}\epsilon^{-aS_s}\right]\epsilon^{-jvS_s}, \tag{47}$$

$$I = \left[\frac{I_sZ_0 - E_s}{2Z_0}\epsilon^{aS_s}\right]\epsilon^{jvS_s} + \left[\frac{I_sZ_0 + E_s}{2Z_o}\epsilon^{-aS_s}\right]\epsilon^{-jvS_s}. \tag{48}$$

An analysis of these two equations shows that the maximum values of voltage and current, as sine functions of time, are the resultant of two vectors, one of which, increasing exponentially with the

distance from the sending end, rotates counter-clockwise, and the other, decreasing exponentially, rotates clockwise, through angular distances directly proportional to the distance from the sending end.

This means that the exponentially decaying maximum values of voltage and current, as sine functions of time, at any point along the line, are the resultant of two sine functions of distance, respectively. In other words, E and I in equations (37) and (38), and in equations (47) and (48), while being maximum values of sine functions of time, are also instantaneous values as sine functions of distance, *i.e.*, each is propagated along the line as a wave with a gradually diminishing amplitude.

The most convenient expressions for calculating the voltage and current at any point along a transmission line, and also for the analysis of special cases of transmission conditions, are equations (37) and (38). Since the effective value of a sine function is the maximum value divided by the $\sqrt{2}$, and since it is conventionally considered as a vector in the study of alternating currents, it is obvious that these equations and also their exponential equivalents, (47) and (48), may be and are ordinarily used for calculating effective values of voltage and current.

If the effective voltage and current along a transmission line are to be calculated from known values at the receiving end of the line, instead of from known values at the sending end, and with distances as measured from that end, equations (37) and (38) become

$$E = E_r \cosh pS_r + I_r Z_0 \sinh pS_r \tag{49}$$

$$I = I_r \cosh pS_r + \frac{E_r}{Z_0} \sinh pS_r \tag{50}$$

where the subscript r indicates quantities at or measured from the receiving end.

21. The Attenuation Constant. — The real component of the propagation constant

$$a = \sqrt[4]{(R^2 + L^2\omega^2)\,(G^2 + C^2\omega^2)} \cos \tfrac{\delta}{2}$$

is called the ***attenuation constant***. Its value, as seen from this equation, depends entirely upon the physical constants of the line. It will be observed from equations (47) and (48) that the quantities ϵ^a and ϵ^{-a} are the per cent increase and decrease, respectively, in voltage or current, either at a point one mile distant from the re-

ceiving end, or one mile distant from the sending end, provided that the line is uniform and of great length.

The quantities ϵ^{aS} and ϵ^{-aS} are called ***attenuation factors***, and they represent the values of voltage or current in per cent of receiving-end or sending-end values, at a point S from either end, respectively.

Since

$$\epsilon^{aS} = \cosh aS - \sinh aS,$$

it is evident that the attenuation factor has the properties of a real hyperbolic angle; its value, therefore, is expressed in terms of hyperbolic radians.

22. The Wave-Length Constant. — As was previously mentioned, the factors ϵ^{jvS} and ϵ^{-jvS} indicate that the components of voltage and current are vector quantities, rotating through an angle, whose value vS, expressed in circular radians, depends upon the line constants, and upon the distance as measured from either end of the line. In other words, each vector rotates through an angle v radians for every mile of distance the voltage or current is propagated. This means also that, for every mile of travel, the voltage and current drop in phase behind their respective sending-end values through an angle of v radians. For this reason the quantity v is frequently called the ***phase constant***.

When the distance S traveled over by a wave is such that $vS = 2\pi$, the voltage and current vectors have rotated through a complete revolution, and the wave has completed one cycle, *i.e.*, it has traveled over a distance equal to one complete wave-length. In such a case if λ = wave-length in miles, we have $S = \lambda$ and $v = 2\pi$, or

$$\lambda = \frac{2\pi}{v}. \tag{53}$$

If the value of v is known, the wave-length can be calculated very easily from the above expression. For this reason v is also called the ***wave-length constant***. Since

$$\epsilon^{jvS} = \cos vS + j \sin vS,$$

it is evident that vS has the properties of a circular angle, therefore its value is expressed in terms of circular radians.

The quantity vS is called ***wave-length factor*** or the ***phase factor***. It represents the angle of lag of the voltage and the current at any point distant S miles from the sending end behind the respective sending-end values.

23. Velocity of Propagation. — Since the frequency f is the number of complete waves or cycles per second, the time in seconds required by a wave to travel over a distance equal to one wave-length is $T = 1/f$. If V = velocity of propagation, we have $V = \lambda/T$. Substituting the value of the wave-length in terms of the wave-length constant, from (53), and the value of T in terms of the frequency, we get

$$V = \frac{2\pi f}{v} = \frac{\omega}{v}, \tag{54}$$

which shows that the velocity of propagation is the ratio of the vector angular velocity to the wave-length constant.

24. Scalar Values of the Attenuation and Wave-Length Constants. — While it may be more convenient to calculate the numerical values of the attenuation constant a and wave-length constant v by equations (44) and (45), respectively, their values may also be obtained as scalar quantities in the following manner. Since

$$p = \sqrt{(R + jL\omega)\ (G + jC\omega)},$$

and since, by (46) $p = a + jv$, it follows that

$$a + jv = \sqrt{(R + jL\omega)\ (G + jC\omega)}.$$

Squaring both sides, we get

$$a^2 + 2\,jva - v^2 = RG - \omega^2 LC + j\omega(RC + LG).$$

Equating the real and the imaginary parts, respectively, on both sides of the equality sign, we obtain the following two equations for the two unknown quantities a and v:

$$a^2 - v^2 = RG - \omega^2 LC,$$
$$2\,av = \omega(RC + LG).$$

From the second of these equations we get

$$a = \frac{\omega}{2\,v}(RC + LG).$$

Substituting this value of a in the first, we obtain

$$\frac{\omega^2}{4\,v^2}(RC + LG)^2 - v^2 = RG - \omega^2 LC$$

or

$$v^4 + (RG - \omega^2 LC)\,v^2 - \frac{\omega^2}{4}(RC + LG)^2 = 0.$$

This is a quadratic equation in v^2; solving it, we have

$$v^2 = \frac{1}{2}\left[\sqrt{(RG - \omega^2 LC)^2 + \omega^2(RC + LG)^2} - (RG - \omega^2 LC)\right].$$

Factoring the quantity under the radical, this equation may be written in the convenient form

$$(55) \qquad v = \sqrt{\frac{1}{2}\left\{\sqrt{(R^2 + L^2\omega^2)\,(G^2 + C^2\omega^2)} - (RG - \omega^2 LC)\right\}}.$$

The scalar value of the attenuation constant may be found in a similar manner. It is

$$(56) \qquad a = \sqrt{\frac{1}{2}\left\{\sqrt{(R^2 + L^2\omega^2)\,(G^2 + C^2\omega^2)} + (RG - \omega^2 LC)\right\}}.$$

25. Characteristic Line Impedance. Line Phase Angle. Surge Impedance. — The quantity

$$Z_0 = \sqrt{\frac{R + jL\omega}{G + jC\omega}},$$

which occurs in general transmission equations is called ***characteristic line impedance***. It is evident from this expression that the characteristic impedance of a line is independent of length of the line, and depends only upon physical constants R, L, G, and C, and upon the frequency of the supply. It is the impedance measured at the sending end of a uniform line, as its length approaches infinity.

The inductive linear impedance of the line, which is the numerator in the preceding expression, is

$$R + jL\omega = \sqrt{R^2 + L^2\omega^2}\,\epsilon^{j\alpha_1},$$

where

$$\alpha_1 = \tan^{-1}\frac{L\omega}{R};$$

and the numerical value of the dielectric admittance per mile of line is

$$G + jC\omega = \sqrt{G^2 + C^2\omega^2}\,\epsilon^{j\alpha_2},$$

where

$$\alpha_2 = \tan^{-1}\frac{C\omega}{G}.$$

Hence, for numerical calculation, the characteristic impedance must be written in the form

$$Z_0 = \sqrt{\frac{\sqrt{(R^2 + L^2\omega)^2}\epsilon^{j\alpha_1}}{\sqrt{(G^2 + C^2\omega)^2}\epsilon^{j\alpha_2}}}.$$

It may be noted that the linear dielectric reactance is always greater than the linear inductive reactance in ordinary lines. Removing the exponential terms from the radical we may write the preceding expression in its final form

$$Z_0 = \sqrt[4]{\frac{R^2 + L^2\omega^2}{G^2 + C^2\omega^2}}\,\epsilon^{j\alpha} \tag{57}$$

where

$$\alpha = \frac{\alpha_1 - \alpha_2}{2}. \tag{58}$$

This angle α may be called the ***characteristic phase angle*** of the line. For reasons given above, the characteristic line impedance will force the current, when the line is in its natural state, *i.e.*, not modified in its physical constants, to lead the voltage by the characteristic phase angle. Hence the angle α is generally a negative angle.

The term Z_0 in equation (57) is a complex quantity. Using the same symbol for the absolute value, the expression may be written in the form

$$Z_0\epsilon^{j\alpha} = \sqrt[4]{\frac{R^2 + L^2\omega^2}{G^2 + C^2\omega^2}}\,(\cos\alpha + j\sin\alpha), \tag{59}$$

in which the real part of this complex quantity is the *energy-consuming component*, and the imaginary part is *the reactive component*, of the characteristic line impedance.

Assuming a hypothetical line with no resistance and no leakage, *i.e.*, $R = 0$ and $G = 0$, the characteristic impedance becomes

$$Z_0 = \sqrt{\frac{L}{C}}. \tag{60}$$

This is called ***the natural impedance*** or the ***surge impedance***. The significance of this concept will appear later.

Again, if the frequency of the supply is zero, *i.e.*, if $\omega = 0$, as is the case with a direct-current supply, the characteristic impedance becomes

$$Z_0 = \sqrt{\frac{R}{G}}, \tag{61}$$

which is the characteristic impedance of a direct-current line.

The fundamental principles developed in the preceding pages, and the practical application of the general transmission equations, will be illustrated by and through the investigation of a transmission line supplied at its sending end with a definite voltage of definite and constant frequency under the following four conditions, to be discussed in the next four chapters:

(1) Length of line approaching infinity.
(2) Line open-circuited at the receiving end.
(3) Line short-circuited at the receiving end.
(4) Line with a receiving apparatus of definite impedance.

26. Calculation of Transmission Constants of No. 10 B. & S. Aerial Line. — The numerical illustrations for these four problems as well as for other problems that will be investigated in succeeding chapters will be based on a line having the following specifications:

Double wire aerial line of No. 10 B. & S. gauge copper wire, with an interaxial distance between the conductors of 12 inches.

The physical constants of this line are:

Resistance per mile of line (loop mile): $R = 10.33$ ohms
Inductance per mile of line: $L = 3.676 \times 10^{-3}$ henries
Capacitance per mile of line: $C = 80.18 \times 10^{-10}$ farads
Leakage conductance per mile of line: $G = 5 \times 10^{-6}$ mhos.

The calculation will also be based on a sinusoidal voltage of 50 volts effective value, of a frequency of 796 cycles per second ($\omega = 5000$) impressed at the sending end of the line. This frequency was chosen because it is commonly taken as the average frequency of telephonic currents.

The phase of this sending-end voltage will be assumed equal to zero, *i.e.*, the vector representing this voltage will be taken as the vector to which all the other vector quantities, unless otherwise stated, will be referred. Under the conditions stated above, the line constants are calculated as follows:

$$\omega = 2\pi f = 2\pi \times 796 = 5000 \text{ circular radians},$$
$$L\omega = 3.676 \times 10^{-3} \times 5000 = 18.38 \text{ ohms},$$
$$C\omega = 80.18 \times 10^{-10} \times 5000 = 40 \times 10^{-6} \text{ mhos},$$

$$\alpha_1 = \tan^{-1}\frac{L\omega}{R} = \tan^{-1}\frac{18.38}{10.33} = 60.7°,$$

$$\alpha_2 = \tan^{-1}\frac{C\omega}{G} = \tan^{-1}\frac{40 \times 10^{-6}}{5 \times 10^{-6}} = 82.9°,$$

$$\alpha = \frac{\alpha_1 - \alpha_2}{2} = \frac{60.7° - 82.9}{2} = -11.1°,$$

$$z = R + jwL = \sqrt{R^2 + L^2\omega^2}\,\epsilon^{j\alpha_1} = 21.1\,\epsilon^{j60.7°} \text{ ohms},$$

$$y = G + jC\omega = \sqrt{G^2 + C^2\omega^2}\,\epsilon^{j\alpha_2} = 40.3 \times 10^{-6}\,\epsilon^{j82.9°},$$

$$Z_0 = \sqrt{\frac{z}{y}} = \sqrt{\frac{21.1\,\epsilon^{j60.7°}}{40.3 \times 10^{-6}\,\epsilon^{j82.9°}}} = 723\,\epsilon^{-j11.1°},$$

$$p = \sqrt{zy} = \sqrt{21.1\,\epsilon^{j60.7°} \times 40.3 \times 10^{-6}\,\epsilon^{j82.9°}} = .0292\,\epsilon^{j71.8°},$$

$$a = p \cos \delta = .0292 \cos 71.8° = .009119 \text{ hyp. radians},$$

$$v = p \sin \delta = .0292 \sin 71.8° = .02774 \text{ circular radians},$$

$$\lambda = \frac{2\pi}{v} = \frac{2\pi}{.02774} = 226.4 \text{ miles},$$

$$V = \frac{\omega}{v} = \frac{5000}{.02774} = 180245 \text{ miles per second.}$$

27. Transmission Constants of Telephone Lines. — Table 1 on the following page gives the electric constants and the calculated transmission constants of the commonly used types of telephone lines at the average telephonic frequency of 796 cycles per second. These transmission constants were calculated as illustrated above for the No. 10. B. & S. aerial line.

28. Fundamental Data for Calculating Voltage and Current Values for the No. 10 B. & S. Line under any Receiving End Conditions. — Table 2 gives the fundamental data obtained from tables of circular trigonometric and hyperbolic functions, and also computed hyperbolic functions of complex angles pertaining to the No. 10 B. & S. line, and which will be referred to in all our illustrative problems pertaining to this line. It will be observed from equation (33), Chapter III, that the argument of the hyperbolic sine of the complex angle $a + jv$ is

$$\beta_2 = \tan^{-1} \frac{\tan vS}{\tanh aS}.$$

The value of this angle as S approaches zero is

$$\beta_2 \tan^{-1} \frac{0}{0},$$

which, obviously, is an indeterminate quantity. It will be recalled, however, that for very small circular and hyperbolic angles, the

TABLE 1

B. & S. Gauge	Style	Electrical Constants per Mile of Line				Transmission Constants; Frequency = 796 Cycles per Second					
		R	C	L	G	Z_0	p	α	v	λ	V
24	Dry core insulated Cable	276	.084 $\times 10^{-6}$	.001	5×10^{-6}	$810\ \epsilon^{-j44.6°}$	$.341\ \epsilon^{j44.7°}$	.242	.239	26.3	20900
22		176	.084 $\times 10^{-6}$	.001	"	$649\ \epsilon^{-j43.9°}$	$.271\ \epsilon^{j45.5°}$	.1897	.1925	32.5	25900
22		176	.070 $\times 10^{-6}$	.001	"	$711\ \epsilon^{-j43.8°}$	$.2482\ \epsilon^{j45.9°}$	.1743	.1767	35.6	28290
19		88	.074 $\times 10^{-6}$	.001	"	$485\ \epsilon^{-j42.5°}$	$.1805\ \epsilon^{j46.1°}$	.127	.128	59	39000
19		88	.054 $\times 10^{-6}$	.001	"	$571\ \epsilon^{-j42.7°}$	$.1543\ \epsilon^{j46.2°}$	.107	.111	46.6	44980
16		44	.074 $\times 10^{-6}$	.001	"	$346\ \epsilon^{-j41.4°}$	$.1281\ \epsilon^{j47.8°}$	.086	.0949	66.1	52660
14		27	.074 $\times 10^{-6}$	.001	"	$270\ \epsilon^{-j39.4°}$	$.0957\ \epsilon^{j43.8°}$	.0777	.0732	85.8	68340
13		22	.074 $\times 10^{-6}$	.001	"	$247\ \epsilon^{-j38.2°}$	$.0914\ \epsilon^{j51°}$	.0575	.0710	88.5	70410
10		11	.074 $\times 10^{-6}$	.001	"	$181\ \epsilon^{-j32.1°}$	$.0669\ \epsilon^{j56.8°}$	.0366	.0559	112.2	89340
14	Bare Cu. clad 47%	59.4	.00795 $\times 10^{-6}$	.00420	"	$1286\ \epsilon^{-j31.8°}$	$.04898\ \epsilon^{j5.°}$	.0308	.0381	165	131200
12		37.2	.00786 $\times 10^{-6}$	.00408	"	$1035\ \epsilon^{-j7°}$	$.04100\ \epsilon^{j55.7°}$	.02305	.03385	185.8	147700
10		23.5	.00820 $\times 10^{-6}$	.00394	"	$861.6\ \epsilon^{-j6.5°}$	$.03558\ \epsilon^{j61.5°}$	.01695	.0312	201.5	160200
9		18.9	.00838 $\times 10^{-6}$	.00390	"	$802\ \epsilon^{-j35.6°}$	$.03381\ \epsilon^{j64.5°}$	.01455	.03055	206	163800
12	N. B. S.	10.26	.00822 $\times 10^{-6}$	.00366	"	$711.7\ \epsilon^{-j11.2°}$	$.02946\ \epsilon^{j71.8°}$	.00916	.0280	224.5	178600
10	B. & S.	10.33	.008018 $\times 10^{-6}$	.003676	"	$723\ \epsilon^{-j11.1°}$	$.0292\ \epsilon^{j71.8°}$	.009119	.02774	226.4	180245
9	B. & S.	8.78	.00838 $\times 10^{-6}$	.00359	"	$688.3\ \epsilon^{-j9.6°}$	$.02904\ \epsilon^{j73.6°}$	.00822	.02785	225.8	179600
8	N. B. S.	6.634	.00857 $\times 10^{-6}$	.003566	"	$663.5\ \epsilon^{-j8°}$	$.0286\ \epsilon^{j77.3°}$	.00628	.02790	225.4	179200
8	B. W. G.	4.08	.00898 $\times 10^{-6}$	.00337	"	$620.1\ \epsilon^{-j3°}$	$.02795\ \epsilon^{j80.1°}$	.0048	.02747	228.7	183016

TABLE 2

S	aS	cosh aS	sinh aS	tanh aS	Radians vS	Degrees vS	sin vS	cos vS	tan vS	$\cosh^2 aS$	$\cos^2 aS$	$\sin^2 vS$
0	0	1.00	0	0	0	0	0	1.0	0	1.0	1.0	0
14.1	.129	1.0087	.1293	.1283	.391	22.5°	.3827	.9239	.4142	1.009	.85	.146
28.2	.258	1.0334	.2608	.2524	.782	45.0°	.707	.707	1.0	1.04	.50	.50
42.3	.387	1.0758	.3967	.3687	1.153	67.5°	.9239	.3927	2.4142	1.16	.146	.850
56.6	.516	1.1361	.5392	.4746	1.57	90°	1.0	0	∞	1.29	0	1.0
70.7	.645	1.2153	.6906	.5682	1.961	112.5°	.9239	$-.3827$	-2.4142	1.48	.146	.850
84.8	.774	1.3153	.8536	.6492	2.352	135°	.707	$-.707$	-1.0	1.70	.500	.500
98.9	.895	1.428	1.0193	.7139	2.742	157.5°	.3827	$-.9239$	$-.4142$	2.04	.85	.146
113.2	1.032	1.5815	1.2251	.7747	3.14	180°	0	-1.0	0	2.5	1.0	0
127.3	1.161	1.7533	1.4399	.8213	3.531	202.5°	$-.3827$	$-.9239$	.4142	3.07	.850	.146
141.4	1.29	1.9540	1.6788	.8591	3.922	225°	$-.707$	$-.707$	1.0	3.82	.500	.500
155.5	1.419	2.1874	1.9455	.8894	4.313	247.5°	$-.9239$	$-.3827$	2.4142	4.75	.146	.850
169.8	1.548	2.4517	2.2447	.9144	4.714	270°	-1.0	0	$-\infty$	6.02	0	1.0
183.9	1.677	2.7682	2.5813	.9325	5.095	292.5°	$-.9239$	.3827	-2.4142	7.68	.146	.850
198.0	1.806	3.1252	2.9608	.9474	5.486	315°	$-.707$	.707	-1.0	9.75	.500	.500
212.1	1.935	3.5347	3.3898	.9591	5.877	337.5°	$-.3827$	.9239	$-.4142$	12.4	.850	.146
226.4	2.064	4.00	3.8753	.8682	6.28	360°	0	1.0	0	16.0	1.0	0

S	aS	$\tanh aS \times \tan vS$	$\tan^{-1}(\tanh aS \over \tan vS)$	$\cosh pS = \sqrt{\cosh^2 aS - \sin^2 vS}$	$\tan vS \over \tanh aS$	$\tan^{-1}\tan vS \over \tanh aS$	$\sinh pS = \sqrt{\cosh^2 aS - \cos^2 rS}$	ctnh pS	tanh pS
0	0	0	0°	$1.0\ \epsilon^{j0°}$	3.0145	71.8°	$0\ \epsilon^{j71.8°}$	$\infty\ \epsilon^{-j71.8°}$	$0\ \epsilon^{j71.8°}$
14.1	.129	.053	3°	$.930\ \epsilon^{j3}$	3.22	72.7°	$.399\ \epsilon^{j72.7}$	$2.32\ \epsilon^{-j69.7°}$	$.431\ \epsilon^{j69.7°}$
28.2	.258	.252	14.2°	$.735\ \epsilon^{j14.2°}$	3.97	75.8°	$.735\ \epsilon^{j75.8°}$	$1.00\ \epsilon^{-j61.8°}$	$1.0\ \epsilon^{j61.8°}$
42.3	.387	.888	41.6°	$.557\ \epsilon^{j41.6°}$	6.5	81.3°	$1.001\ \epsilon^{j81.3°}$	$.557\ \epsilon^{-j39.7°}$	$1.795\ \epsilon^{j39.7°}$
56.6	.516	∞	90°	$.538\ \epsilon^{j90°}$	∞	90°	$1.132\ \epsilon^{j90°}$	$.476\ \epsilon^{-j°}$	$2.11\ \epsilon^{j0°}$
70.7	.645	-1.37	126.1°	$.795\ \epsilon^{j126.1°}$	-4.24	103.3°	$1.11\ \epsilon^{j103.3°}$	$.707\ \epsilon^{j22.8°}$	$1.414\ \epsilon^{-j22.8°}$
84.8	.746	$-.643$	147.3°	$1.1\ \epsilon^{j147.3°}$	-1.555	122.8°	$1.1\ \epsilon^{j122.8°}$	$1.00\ \epsilon^{j24.5°}$	$1.0\ \epsilon^{-j24.5}$
98.9	.895	$-.320$	162.5°	$1.37\ \epsilon^{j162.5°}$	$-.578$	150°	$1.089\ \epsilon^{j150°}$	$1.258\ \epsilon^{j12.5°}$	$.796\ \epsilon^{-j12.5°}$
113.2	1.032	0	180°	$1.58\ \epsilon^{j180°}$	0	180°	$1.225\ \epsilon^{j150°}$	$1.29\ \epsilon^{j0°}$	$.775\ \epsilon^{j0°}$
127.3	1.161	34	198.8°	$1.68\ \epsilon^{j198.8°}$	.503	206.7°	$1.49\ \epsilon^{j206.7°}$	$1.129\ \epsilon^{-j7.9°}$	$.897\ \epsilon^{j7.9°}$
141.4	1.29	.859	220.6°	$1.82\ \epsilon^{j220.6°}$	1.165	229.3°	$1.82\ \epsilon^{j229.3°}$	$1.00\ \epsilon^{-j8.7°}$	$1.0\ \epsilon^{j8.7°}$
155.5	1.419	2.14	245°	$1.97\ \epsilon^{j245°}$	2.72	249.8°	$2.14\ \epsilon^{j249.8°}$	$.922\ \epsilon^{-j4.8°}$	$1.085\ \epsilon^{j4.8°}$
169.8	1.548	∞	270°	$2.26\ \epsilon^{j270°}$	∞	270°	$2.45\ \epsilon^{j270°}$	$.922\ \epsilon^{j0°}$	$1.085\ \epsilon^{j0°}$
183.9	1.677	-2.225	293°	$2.61\ \epsilon^{j293°}$	-2.59	291.1	$2.72\ \epsilon^{j241.1°}$	$.961\ \epsilon^{j1.9°}$	$1.04\ \epsilon^{-j1.9°}$
198.0	1.806	$-.947$	316.6°	$3.04\ \epsilon^{j316.6°}$	-1.056	313.4°	$3.04\ \epsilon^{j313.4°}$	$1.0\ \epsilon^{j3.2°}$	$1.0\ \epsilon^{-j3.2°}$
212.1	1.935	$-.396$	338.6°	$3.5\ \epsilon^{j338.6°}$	$-.432$	336.6°	$3.39\ \epsilon^{j336.6°}$	$1.032\ \epsilon^{j2°}$	$.967\ \epsilon^{-j2°}$
226.4	2.064	0	360°	$4.0\ \epsilon^{j360°}$	0	360°	$3.87\ \epsilon^{j360°}$	$1.032\ \epsilon^{j0°}$	$.967\ \epsilon^{j0°}$

tangent of the angle approaches in value the angle itself, when expressed in radians, *i.e.*, $\tan vS = vS$, and $\tanh aS = aS$, when vS and aS are respectively very small. It follows, therefore, that when S approaches zero, the argument of the hyperbolic sine of the complex angle $a + jv$ may be written in the form

$$\beta_2 = \tan^{-1} \frac{vS}{aS} = \tan^{-1} \frac{v}{a}.$$

Hence, the limiting value of this argument for the No. 10 B. & S. line is

$$\beta_2 = \tan^{-1} \frac{.02774}{.009119} = \mathbf{71.8°}.$$

CHAPTER V

THE INFINITELY LONG LINE

The study of the propagation of electric waves along a line of infinite length, while it appears to be of only academic interest, will be found to be of great importance from a practical point of view, since an infinite line, presenting, as will be seen, a constant impedance at any point along its length, gives a very simple mode of propagation. In fact, it is so simple that any other line of any length having equal electrical constants per mile, but with any receiving-end conditions, may be expressed most conveniently in terms of the infinitely long line.

29. Line Impedance and Sending-End Current. — Since both voltage and current attenuate as their respective waves travel along the line, both current and voltage will approach zero when the line approaches infinity. Accordingly, when the distance from the sending end is $S_s = \infty$, the two general transmission equations (37) and (38), Chapter IV, may be equated to zero. Equation (37) may therefore be written in the form

$$E_s \cosh pS = I_s Z_0 \sinh pS_s,$$

or

$$(1) \qquad E_s = I_s Z_0 \frac{\sinh pS_s}{\cosh pS_s} = I_s Z_0 \tanh pS.$$

But when $S_s = \infty$, we have $\tanh pS_s = 1$. It follows therefore that for an infinitely long line

$$(2) \qquad E_s = I_s Z_0.$$

Inserting the phase angle of Z_0, and solving for the sending-end current, we get

$$(3) \qquad I_s = \frac{E_s}{Z_0} \epsilon^{-j\alpha}.$$

The physical significance of the last equation is that the impedance offered by an infinitely long line at its sending end, or what is generally called the ***sending-end impedance,*** is equal to the *characteristic line impedance* (§ 25). In other words, the im-

pedance of an infinite line at any point along its length is the same and is equal to the characteristic impedance.

30. Voltage and Current at any Point along an Infinite Line. — The potential at any point along a line by (37), Chapter IV, is

$$E = E_s \cosh pS_s - I_s Z_0 \sinh pS_s.$$

Substituting for I_s its value from (2), we get for the potential at any point of an infinite line

$$E = E_s (\cosh pS_s - \sinh pS_s). \tag{4}$$

Similarly, substituting the value of E_s from (2) in the general equation for the current along a transmission line, we find

$$I = I_s (\cosh pS_s - \sinh pS_s). \tag{5}$$

Since we have

$$\epsilon^{-pS_s} = \cosh pS_s - \sinh pS_s,$$

it follows that the voltage and the current at any point along an infinite line may be written in the forms

$$E = E_s \epsilon^{-pS_s}, \tag{6}$$

$$I = I_s \epsilon^{-pS_s}. \tag{7}$$

Substituting for the propagation constant p its value in terms of the attenuation and wave-length constant, $a + jv$, we may write the last two equations in the final forms

$$E = E_s \epsilon^{-(a+jv)S_s}, \tag{8}$$

$$I = I_s \epsilon^{-(a+jv)S_s}. \tag{9}$$

These equations show that with an effective voltage of E_s , impressed at the sending end of an infinite line, resulting in a current at the sending end equal to I_s, both voltage and current will decrease to values $E_s \epsilon^{-aS_s}$ and $I_s \epsilon^{-aS_s}$, respectively, at a distance S_s from the sending end.

The equations also indicate that both voltage and current will lag behind their respective sending-end values by a phase angle equal to vS_s radians, or v radians per mile of line. For this reason the constant v is sometimes called the ***phase constant.***

If we substitute in equation (9) the value of the sending-end current as obtained in equation (3), the current at a distance S_s from the sending end of an infinite line, in terms of the sending voltage, is

$$I = \left[\frac{E_s}{Z_0} \epsilon^{-aS_s}\right] \epsilon^{-j(\alpha + vS_s)} \tag{10}$$

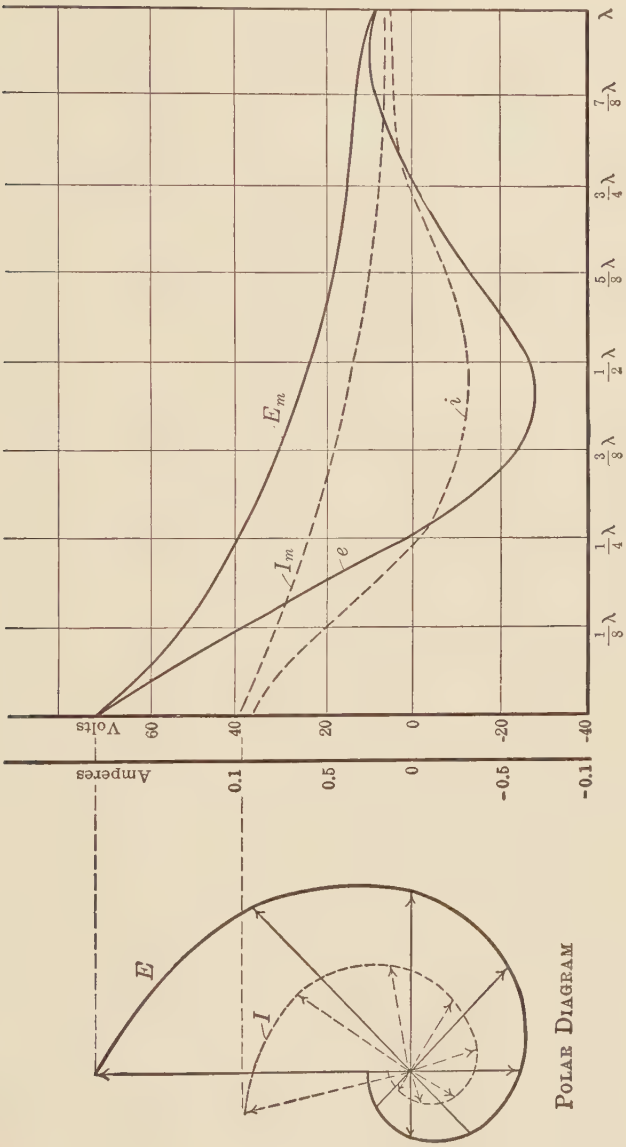

Fig. 7. — Propagation of Voltage and Current. — Infinite Line

This equation shows that the line impedance, which is equal to the characteristic line impedance for this particular line, changes the phase of the current vector with respect to the voltage, at any point along the line, by a leading angle equal to the characteristic line phase angle, assuming the conventional positive direction of vectors.

Table 3 gives the calculated values of the attenuation factor aS_s, the per cent attenuation ϵ^{-aS_s}, the phase angle with respect to the sending-end voltage, assumed 50 volts effective, and the effective values of voltage and current at increments of line equal to one-eighth of a wave-length, for a portion of an infinite No. 10 B. & S. line equal to one complete wave-length.

TABLE 3

S_s	aS_s	ϵ^{-aS_s}	$E\epsilon^{-jvS_s}$	$I\epsilon^{j(11.1^\circ - vS_s)}$
0	0	1.0	50.00 ϵ^{-j0°	.0692 $\epsilon^{j11.1^\circ}$
28.2	.258	.7727	38.8 ϵ^{-j45°	.0537 $\epsilon^{-j33.9^\circ}$
56.6	.516	.5969	29.8 ϵ^{-j90°	.0413 $\epsilon^{-j78.9^\circ}$
84.8	.774	.4621	23.2 ϵ^{-j135°	.0321 $\epsilon^{-j123.9^\circ}$
113.2	1.032	.3563	17.8 ϵ^{-j180°	.0247 $\epsilon^{-j168.9^\circ}$
141.4	1.29	.2753	13.75 ϵ^{-j225°	.01905 $\epsilon^{-j213.9^\circ}$
169.8	1.548	.2127	10.6 ϵ^{-j270°	.01467 $\epsilon^{-j258.9^\circ}$
198.	1.806	.1641	8.2 ϵ^{-j315°	.01135 $\epsilon^{-j303.9^\circ}$
226.4	2.064	.1275	6.37 ϵ^{-j360°	.00883 $\epsilon^{-j348.9^\circ}$

Curve E, Fig. 7, shows the variation of maximum instantaneous voltage, and curve I the variation of maximum instantaneous current values with distance as measured from the sending end. Curves e and i show the variation of space instantaneous values of voltage and current values with distance from the sending end.

The last two curves were obtained graphically from the polar diagram shown on the left, in Fig. 7, and the maximum values plotted were calculated by multiplying the effective values given in the above table by $\sqrt{2}$.

CHAPTER VI

THE OPEN-CIRCUITED LINE

31. Sending-End Impedance. — The sending-end impedance is the impedance offered at the sending end of the line to the propagation of the current. This definition implies that the sending-end impedance is the ratio of the sending-end voltage to the sending-end current:

$$Z_s = \frac{E_s}{I_s}. \tag{1}$$

Since the line to be investigated in this chapter is open-circuited at its far end, it is obvious that the current at that end must be equal to zero. For this particular condition, then, the general transmission equation for current ((38) Chapter IV) may be written in the form

$$0 = I_s \cosh pS - \frac{E_s}{Z_0} \sinh pS,$$

where S = length of line. Solving this equation for Z_0, we get

$$Z_0 = \frac{E_s}{I_s} \frac{\sinh pS}{\cosh pS},$$

or, by equation (1),

$$Z_0 = Z_s \tanh pS. \tag{2}$$

Hence the equation

$$Z_s = \frac{Z_0}{\tanh pS} = Z_0 \frac{\cosh (a + jv)S}{\sinh (a + jv)S} \tag{3}$$

gives the value of the sending-end impedance in terms of the characteristic line impedance, the transmission constants of the line, and the length of line.

For calculating the numerical value of the sending-end impedance, exponential forms must be used for Z_0 and for the hyperbolic sine and cosine of the complex angle $a + jv$. This gives

$$Z_s = Z_0 \epsilon^{j\alpha} \frac{\sqrt{\cosh^2 aS - \sin^2 vS}\, \epsilon^{j\beta_1}}{\sqrt{\cosh^2 aS - \cos^2 vS}\, \epsilon^{j\beta_2}} \tag{4}$$

where

$$\beta_1 = \tan^{-1} \tanh aS \tan vS, \tag{5}$$

and

$$\beta_2 = \tan^{-1} \frac{\tan vS}{\tanh aS}. \tag{6}$$

If we let $\beta_1 - \beta_2 = \beta$, we may write the sending-end impedance in the final form

$$Z_s = Z_0 \sqrt{\frac{\cosh^2 aS - \sin^2 vS}{\cosh^2 aS - \cos^2 vS}}\, \epsilon^{j(\beta+\alpha)}. \tag{7}$$

This angle $(\beta + \alpha)$ is the phase displacement that the sending-end impedance will produce between the sending-end voltage and the sending-end current.

32. Sending-End Current. — Since the sending-end current is the ratio of the sending-end voltage to the sending-end impedance, whose value is given by equation (7), it follows that the sending-end current is

$$I_s = \frac{E_s}{Z_s} \epsilon^{-j(\beta+\alpha)}. \tag{8}$$

33. Voltage at the Open-Circuited End. — Using the general transmission equation, the expression for the voltage at the open-circuited end of a line of length S, is

$$E_r = E_s \cosh pS - I_s Z_0 \sinh pS,$$

where the subscript r indicates quantities at or quantities measured from the receiving end. Using the relation

$$I_s = \frac{E_s}{Z_s},$$

and using the value of Z_s from (3), we find

$$I_s = \frac{E_s}{Z_0} \frac{\sinh pS}{\cosh pS}.$$

Substituting this value of I_s in the above expression for the voltage, we get

$$E_r = E_s \cosh ps - E_s \frac{\sinh^2 pS}{\cosh pS} = E_s \frac{\cosh^2 ps - \sinh^2 pS}{\cosh pS},$$

or

$$E_r = \frac{E_s}{\cosh ps} = \frac{E_s}{\cosh (a + jv)S}. \tag{9}$$

For purposes of calculation, the exponential form must be used for the hyperbolic cosine, and the expression for the receiving-end voltage becomes

$$E_r = \frac{E_s}{\sqrt{\cosh^2 aS - \sin^2 vS}} \epsilon^{-j\beta_1}. \tag{10}$$

where β_1 has the value given in equation (5).

An analysis of equation (10), by means of which the open-circuited end-voltage of any line may be calculated, reveals some very interesting facts. It will be noted that the denominator of the expression is the difference between a hyperbolic term whose value increases with the length of line, and a circular trigonometric member, whose value changes periodically with the line-length from zero to a maximum of 1 and back to zero. As the line is increased in length, *i.e.*, as S is made larger and larger, the denominator, while increasing hyperbolically, passes through periodic maxima and minima, as the circular trigonometric member (on increasing the length of line) passes through zero and unity, respectively. It is also evident that these maxima and minima become less and less apparent, the larger the value of S, since the values of the circular trigonometric term become small in comparison with those of the hyperbolic term. In other words, the voltage at the open end of the line is a minimum when the line length is such that $\sin vS_1 = 0$, *i.e.*, when

$$vS_1 = 2n\frac{\pi}{2}; \tag{11}$$

and it will be a maximum when the line length is such that $\sin vS_2 = 1$, *i.e.*, when

$$vS_2 = (2n - 1)\frac{\pi}{2} \tag{12}$$

where S_1 = length of line for a minimum receiving-end voltage,
S_2 = length of line for a maximum receiving-end voltage,
n = any integer.

If we substitute for the wave-length constant v its value in terms of the wave-length ((53) Chapter IV), (11) and (12) may be written, respectively, in the forms

$$S_1 = 2n\frac{\lambda}{4}, \tag{13}$$

$$S_2 = (2n - 1)\frac{\lambda}{4}. \tag{14}$$

These equations show that with a voltage of definite frequency impressed at the sending end of the line, the condition that a minimum voltage occur at the far end of an open-circuited line, assumed of variable length, is that line-length must be an even number of quarter wave-lengths. Similarly, the condition that a maximum voltage occur at the far end of the line is that the line-length must be equal to an odd number of quarter wave-lengths.

The dotted curve in Fig. 14 shows graphically the variation of the open-end voltage with line-length for the No. 10 B. & S. line, with 50 volts at a frequency of 796 cycles impressed at its sending end. The values of line-length against which the voltage is plotted is expressed in terms of fraction of wave-lengths. It will be observed that the maximum that occurs when the length of line is equal to one quarter of a wave-length, is almost double the sending-end voltage, while the maximum that occurs when the line is three-quarters of a wave-length is barely noticeable, indicating that the trigonometric term in equation (10), applied to the No. 10 B. & S. line, becomes negligible when the length of the line is more than three quarters of a wave-length.

34. Resonance Effects and Natural Line Frequency. — The occurrence of a maximum voltage at the open end of an open-circuited line, when the line is an odd number of quarter wave-lengths, may be explained by substituting in (12) the scalar value of the wave-length constant as given by (55), Chapter IV, and solving for the frequency, this gives

$$\sqrt{\frac{1}{2}\left\{\sqrt{(R^2+L^2\omega^2)\,(G^2+C^2\omega^2)}-(RG-\omega^2LC)\right\}}=\frac{(2n-1)\,\pi}{2\,S_2},$$

whence

$$(15)\qquad \omega=\frac{(2n-1)\,\pi}{2\,S_2}\sqrt{\frac{(2n-1)^2\,\pi^2+4\,S_2RG}{S_2(RC+LG)^2+(2n-1)^2\pi^2LC}},$$

and the frequency is

$$(16)\qquad f=\frac{2n-1}{4\,S_2}\sqrt{\frac{(2n-1)^2\pi^2+4\,S_2RG}{S_2(RC+LG)^2+(2n-1)^2\pi^2LC}}.$$

This expression gives the value of the frequency for maximum voltage at the open end of the line, in terms of line constants and length of line.

Assuming a hypothetical line with no resistance and no leakage conductance, *i.e.*, taking R and G equal to zero, the expression is reduced to the simple form

$$f = \frac{2n-1}{4S_2}\sqrt{\frac{1}{LC}}. \tag{17}$$

But the inductance per mile of an aerial copper line is

$$L = 2\left(.000080742 + .00074113 \log_{10}\frac{2D}{d}\right) \text{ henries}, \tag{18}$$

and the mutual capacitance per mile of line is

$$C = \frac{.01941}{\log_{10}\frac{2D}{d}} \times 10^{-6} \text{ farads}, \tag{19}$$

where D denotes the interaxial distance between the conductors, and d denotes the diameter of the conductors in the same units in which D is measured. Neglecting the first term in the expression for the inductance, and taking the square root of the reciprocal of the product of the inductance and the capacitance per mile of line from (18) and (19), we get

$$\sqrt{\frac{1}{LC}} = 186300. \tag{20}$$

Using this value in connection with (17), we find

$$\frac{4fS_2}{2n-1} = 186300. \tag{21}$$

It follows that a maximum potential occurs at the end of the line when the relation between the frequency and length of line is that given by (21), if we assume that the resistance and the leakage are negligible. When $n = 1$, *i.e.*, when $S_2 = \lambda/4$, (21) becomes:

$$4S_2 f = \lambda f = V = 186300.$$

Hence the condition that a maximum potential occur at the end of an open-circuited line is that the product of the frequency and line-length must be equal to one-fourth of the constant 186300, *i.e.*, to one-fourth of the velocity of light. It follows also that the maximum velocity of electric wave propagation is equal to that of light.

Evidently, a line that has appreciable resistance and leakage conductance will permit the propagation of electric waves with

velocities less than that of light, depending upon both the resistance and the leakage.

When $n = 2$, *i.e.*, when $S_2 = 3\,\lambda/4$, the condition that maximum potential occur at the end of the line is that the product of the frequency and line-length must be equal to three-fourths of the velocity of light. Moreover, in such a case there is a maximum of the first order at a distance from the sending end equal to one-fourth of a wave-length, and a maximum of the second order at the end of the line. It is also obvious, from what has just been said, that a minimum voltage will occur midway between the two maxima, *i.e.*, at a distance from the sending end equal to one-half of a wave-length.

The voltage curves, Figs. 9 and 12, are drawn for lengths of line one-quarter and three-quarters of a wave-length, respectively. They illustrate the results obtained in the above discussion.

Equation (21) suggests two ways in which conditions of maximum potential at the end of an open-circuited line may be brought about.

(*a*) For any definite frequency of the supply, the length of the line may be changed so that equation (21) is satisfied, *i.e.*,

$$S_2 = \frac{(2n-1)\,186300}{4f_1}. \tag{22}$$

Thus, if a maximum voltage is to occur only at the end of the line, then $n = 1$, and

$$S_2 = \frac{186300}{4f_1}.$$

Obviously, the length of the line is made, in this particular case, equal to one-quarter of a wave-length.

If two maxima are to occur, one at the open end of the line, and the other at one-third of the line-length from the sending end, then $n = 2$, and

$$S_2 = \frac{3 \times 186300}{4f_1}$$

In this case, the line is made equal to three-quarters of a wave-length.

(*b*) To bring about conditions of maximum potential, a more practicable plan is to make the frequency of the supply satisfy equation (21) for any definite length of line S_2.

Thus, if a maximum is to occur only at the end of the line, $n = 1$, and

$$f_1 = \frac{186300}{4\,S_2}, \tag{23}$$

so that the frequency is such that the line is one-quarter of the wave-length corresponding to that frequency.

If two maxima are to occur, one at the open end of the line, and the other at one-third the line-length as measured from the sending end, $n = 2$ and

$$f_3 = \frac{3 \times 186300}{4\,S_2}.$$

Clearly the frequency in this particular case must be three times the frequency for one-quarter of a wave-length, so that the wave-length is one-third of that in the first case. As the length of the line is supposed to be unchanged, it is now equal to three-fourths of the wave-length corresponding to the frequency f_3.

For practical lines, with appreciable resistance and leakage, the values of line-length and frequency for maximum potential at the open-circuited end will differ from those obtained above, the difference depending upon the amount of resistance and leakage per mile.

To illustrate the above discussion, consider a voltage of a frequency of 300 cycles per second, impressed at the sending end of an open-circuited line. That the voltage at the open end be a maximum, the line-length must be

$$S_2 = \frac{186300}{4 \times 300} = 155.2 \text{ miles.}$$

The wave-length will be

$$\lambda = 4 \times 155.2 = 620.8 \text{ miles.}$$

If two maxima are to occur, the length of the line must be

$$S_2 = \frac{3 \times 186300}{4 \times 300} = 465.6 \text{ miles.}$$

In this case a maximum of the first order occurs at a distance 155.2 miles from the sending end, and a maximum of the second order occurs at the end of the line. The line is three-quarters of a wave-length long, the wave-length being 620.8 miles, as before.

As previously stated, these values of line-length will differ slightly

from those obtained above, if line resistance and leakage are taken into account.

If the line is of a definite length, say 200 miles, and a maximum voltage is to occur at its end, assumed open-circuited, the frequency of the supply must be

$$f_1 = \frac{186300}{4 \times 200} = 232.8 \text{ cycles per sec.}$$

In this case, the line-length is equal to one-quarter of the wave-length corresponding to this frequency, and the wave-length is $4 \times 200 = 800$ miles.

If two maxima of potential are to occur, one at the open end of the line and the other at one-third line-length measured from the sending end, the frequency of the supplied voltage must be

$$f_3 = \frac{3 \times 186300}{4 \times 200} = 697.6 \text{ cycles per sec.}$$

The length of the line is then three-quarters of a wave-length, the wave-length being 266.7 miles, or one-third of the wave-length obtained in the previous case.

The phenomena discussed above are known as ***resonance***, and the frequencies at which they occur are called ***resonant frequencies***. With line resistance and leakage conductance negligibly small, the resonant frequency is that at which the line capacitance and line inductance discharge harmonically through one another, in an oscillatory self-sustained manner through the surge impedance $\sqrt{L/C}$ of the line. The oscillations, once started, will continue indefinitely, even with the impressed electromotive force removed from the sending end of the line. With line resistance and leakage of appreciable values, the line capacitance and inductance discharge through one another against the characteristic line impedance, and the oscillations decay hyperbolically, with the removal of the impressed voltage.

35. Equivalent Line Inductance and Capacitance. — By equation (17), when $n = 1$ the fundamental frequency that would produce resonance, was seen to be

$$f_1 = \frac{1}{4 S_2 \sqrt{LC}},$$

or

$$f_1 = \frac{1}{4 \sqrt{L_t C_t}}, \tag{24}$$

where

$$L_t = S_2 L = \text{total line inductance}$$

and

$$C_t = S_2 C = \text{total line capacitance.}$$

Assuming a circuit having an inductance and a capacitance equal to those of the line under consideration, but in a concentrated form instead of distributed as they are in the actual line, we see that the condition for resonance is

$$L_t \omega = \frac{1}{C_t \omega},$$

and that the frequency for resonance, obtained by solving this equation for ω and dividing by 2π, is

$$f_2 = \frac{1}{2\pi\sqrt{L_t C_t}}, \tag{25}$$

where f_2 is the resonant frequency for the circuit with concentrated inductance and capacitance. Dividing equation (24) by equation (25), we get

$$\frac{f_1}{f_2} = \frac{\pi}{2} = 1.57 \tag{26}$$

This shows that the natural or resonant frequency of a circuit with distributed inductance and capacitance, as they occur, for instance, in a transmission line, is 57 per cent larger than the resonant frequency of a circuit having equal inductance and capacitance as the line but in a lumped form. In other words, a circuit with concentrated inductance L_1 and concentrated capacitance C_1, is equivalent to a line at the same frequency when

$$\frac{1}{4\sqrt{L_t C_t}} = \frac{1}{2\pi\sqrt{L_1 C_1}},$$

or

$$\frac{\sqrt{L_1 C_1}}{\sqrt{L_t C_t}} = \frac{4}{2\pi},$$

or

$$L_1 = \frac{2}{\pi} L_t = .637\, L_t, \tag{27}$$

and

$$C_1 = \frac{2}{\pi} C_t = .637\, C_t. \tag{28}$$

It follows from the above that the concentrated inductance and capacitance of a circuit equivalent to a line with uniformly dis-

tributed constants must be 63.7 per cent of the inductance and capacitance, respectively, of the latter line. It may also be deduced that determined amounts of capacitance and inductance distributed uniformly have 57 per cent more influence on line resonance, than like amounts of inductance and capacitance concentrated.

36. The Ferranti Effect. — In the discussion of the expression for the end voltage of an open-circuited line

$$E_r = \frac{E_s}{\sqrt{\cosh^2 aS - \sin^2 vS}}\,\epsilon^{-j\beta_1},$$

leading to the results obtained in the preceding pages of this Chapter, no attention was given to the possible values of the hyperbolic and trigonometric terms relative to one another. An analysis of the relation between these terms will lead us to a determination of the conditions favorable for potentials at the open ends of lines in excess of the voltages impressed at the sending end. This discussion furnishes also an explanation of this phenomenon, which is generally called the ***Ferranti effect.***

Considering the above expression from the point of view just stated, it will be noted first, that since $S = 0$ at the sending end of the line, the hyperbolic term $\cosh^2 aS = 1$, the trigonometric term $\sin^2 vS = 0$, whence $E_r = E_s$, which is obvious.

As the line-length is increased, however, the hyperbolic term increases above 1, while the trigonometric term varies between zero and unity, its maximum never exceeding 1. If the value of the attenuation constant a and of the wave-length constant v are such that for a definite length of line

$$\cosh^2 aS - \sinh^2 vS < 1, \tag{29}$$

then the denominator of the expression for E_r is less than 1, and the voltage at the open-circuited end is larger than that at the sending end. Such a condition evidently obtains when $\cosh^2 aS$ is less than 2, because $\sin^2 vS$ can never be larger than 1. For the limiting condition, when

$$\cosh^2 aS = 2 \tag{30}$$

and

$$\sin^2 vS = 1, \tag{31}$$

the denominator of the equation for E_r is unity, and the end voltage is again equal to the sending-end voltage.

With definite values of the attenuation constant a and the wave-length constant v for all values of S such that

$$\cosh^2 aS - \sin^2 vS > 1,$$

the denominator of the equation will be larger than 1, and the open-circuit-end voltage will be smaller than the sending-end voltage.

It follows, from what has just been said, that if the Ferranti effect is to occur at the end of a line, the hyperbolic term in the equation must be less than 2. Moreover, with

$$\cosh^2 aS < 2,$$

the denominator will be a minimum when the length of the line is such that

$$\sin^2 vS = 1,$$

resulting in a maximum value of E_r, *i.e.*, the maximum Ferranti effect occurs when the line is in resonant condition, with length of line equal to one-quarter of a wave-length.

To determine whether any line is subject to the above-mentioned favorable conditions for the Ferranti effect, consider the limiting conditions, namely when the open-end voltage is equal to the sending-end voltage. We have

$$\cosh^2 aS = 2, \qquad \text{and} \quad \sin^2 vS = 1$$

or

$$\cosh aS = 1.4142 \quad \text{and} \quad \sin vS = \pm 1,$$

whence

$$aS = .8814 \tag{32}$$

and

$$vS = (2n - 1)\frac{\pi}{2}. \tag{33}$$

Dividing (33) by (32), we get

$$\frac{v}{a} = \frac{(2n - 1)\frac{\pi}{2}}{.8814}. \tag{34}$$

If the line is in the one-quarter of a wave-length resonant condition, $n = 1$, and the last expression becomes

$$\frac{v}{a} = 1.7828 \tag{35}$$

This result indicates that any open-circuited line is subject to the Ferranti effect when the ratio of its wave-length constant to its

attenuation constant is more than 1.7828, *i.e.*, when the attenuation constant is less than 56.09 per cent of the wave-length constant. If this condition is satisfied, the Ferranti effect will occur for all values of line-length between the limits of $S = 0$ and the value of S that will make

$$\cosh^2 aS - \sin^2 vS = 1,$$

that is,

$$-\sin^2 vS = 1 - \cosh^2 aS,$$

or

$$-\sin^2 vS = -\sinh^2 aS,$$

or

$$\sin vS = \sinh aS. \tag{36}$$

This limiting value of S is obviously the intersection of the curves $\sin vS$ and $\sinh aS$ plotted against line length S, as shown in Fig. 8 for the illustrative No. 10 B. & S. line. No Ferranti effect

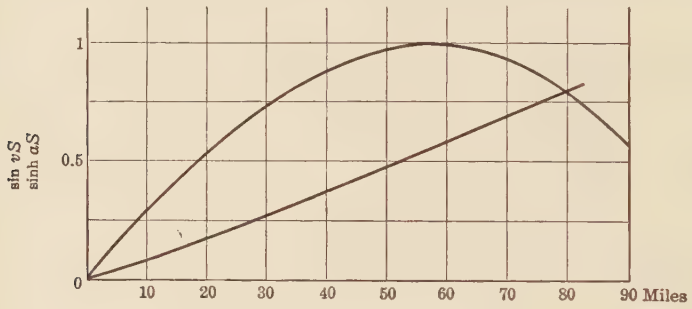

FIG. 8

can occur for lines longer than that given by equation (36). The maximum Ferranti effect, as previously explained, will occur when the length of the line is just one-quarter of a wave-length.

37. Numerical Illustration. — To illustrate the conclusions reached at in the preceding pages, consider our illustrative No. 10 B. & S. line. The attenuation constant and the wave-length constant, as calculated in Chapter IV, are

$$a = .009119, \quad v = .02774$$

Since their ratio

$$\frac{a}{v} = .328$$

is less than the limiting value of .5609, this line is subject to the Ferranti effect.

The limiting length of line for the Ferranti effect as given by Fig. 8 is 80 miles. For any length of line between 0 and 80 miles, the voltage at the end of the line, when open-circuited, will be greater than that impressed at the sending end. See the dotted curve in Fig. 14.

38. Voltage at any Point along an Open-Circuited Line. — By equation (49), Chapter IV, the voltage at any point along a transmission line in terms of receiving-end quantities is:

$$E = E_r \cosh pS_r + I_r Z_0 \sinh pS_r$$

where E_r and I_r are the receiving-end voltage and current, respectively, and S_r is the distance measured from the receiving end.

Since the line is assumed to be open-circuited at the receiving end, the current at that end is obviously zero; hence the above equation may be written in the form

$$E = E_r \cosh pS_r = E_r \cosh (a + jv)\, S_r. \tag{37}$$

Using exponential forms for E_r from (10), and also for the hyperbolic cosine of the complex angle, we get

$$E = E_r \epsilon^{-j\beta_1} \sqrt{\cosh^2 aS_r - \sin^2 vS_r}\, \epsilon^{j\beta_3}, \tag{38}$$

where

$$\beta_3 = \tan^{-1} (\tanh aS_r \tan vS_r).$$

In equation (38) the receiving-end voltage $E_r \epsilon^{-j\beta_1}$ is fixed, its value depending upon the sending-end voltage, length of line, and line constants at the frequency of supply. The variable quantity is the distance along the line S_r, as measured from the open-circuited end. At the open-circuited end where $S_r = 0$, it is evident that $\cosh^2 aS_r = 1$ and $\sin^2 vS_r = 0$. Hence, at the open-circuited end,

$$E = E_r \epsilon^{-j\beta_1}.$$

As the distance S_r is increased toward the sending end, $\cosh^2 aS_r$ increases above the value of 1, and the circular trigonometric term $\sin^2 vS_r$ increases periodically from zero to 1 and back to zero. When S_r has a value that makes the trigonometric term equal to unity, the radical in equation (38) is a minimum; hence the potential at such points are minima. But since $\sin vS_r = 1$ when

$$vS_r = (2n - 1)\frac{\pi}{2},$$

it follows that, beginning at the open end of the line and measuring

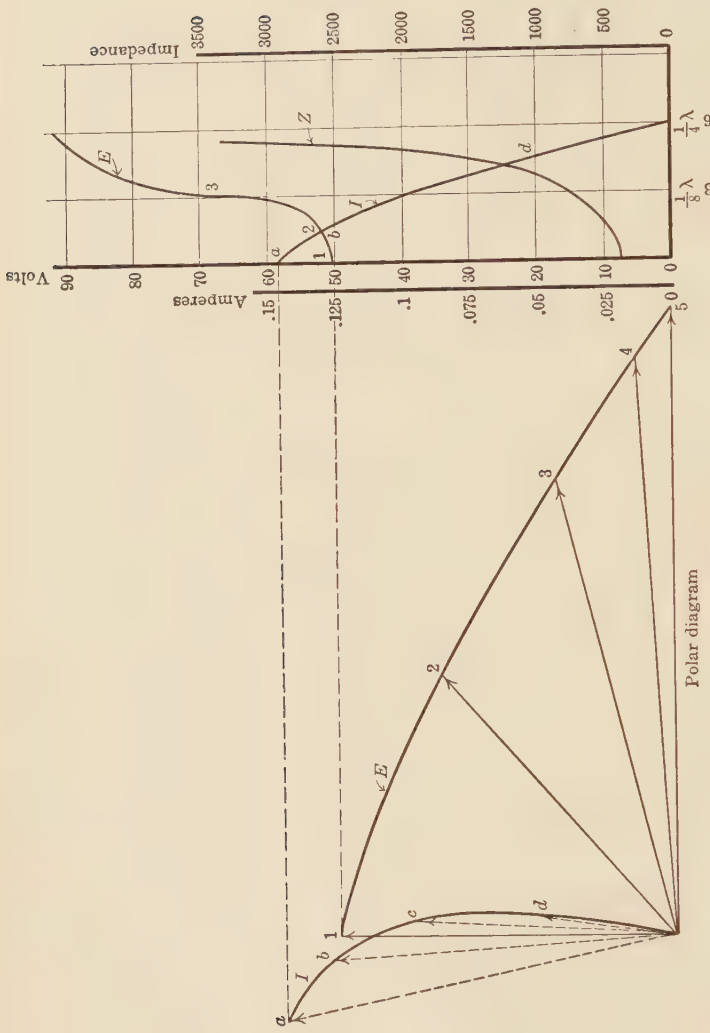

FIG. 9. — Variation of E, I, and Z with Distance. Open-circuited No. 10 B. & S. Line. S = 56.6 Miles

distances from that end, the voltage will pass through minima at every odd quarter wave-length and through maxima at every even quarter wave-length.

Since $\cosh^2 aS_r$ increases rather rapidly as S_r increases, the minimum voltage, which occurs at the first quarter of a wave-length from the open end of the line, is more marked than those

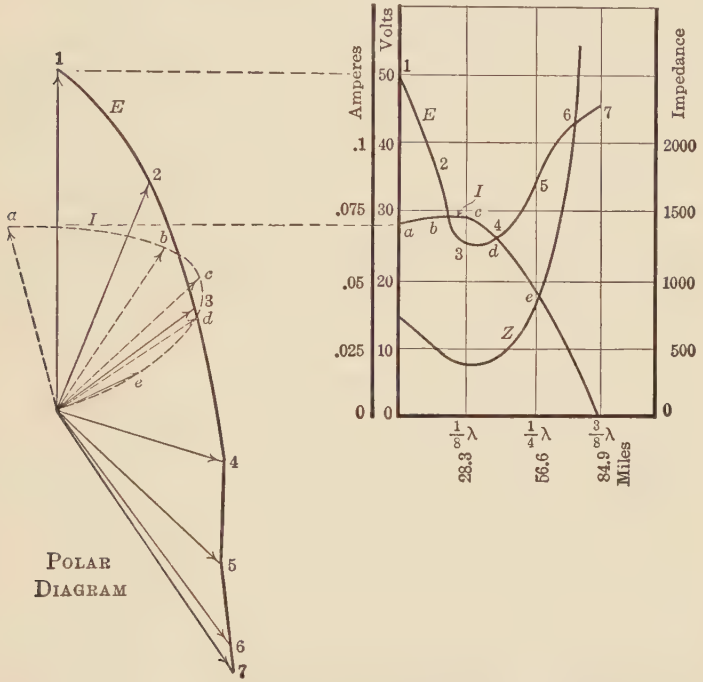

Fig. 10. — Variation of E, I, and Z with Distance. Open-circuited No. 10 B. & S. Line. S = 84.8 Miles

that occur at subsequent odd quarter wave-lengths. Table 4, page 68, gives the calculated values of potential at points one-sixteenth of a wave-length apart for the No. 10 B. & S. line, for lengths of line as indicated, and with 50 volts at a frequency of 796 cycles per second impressed at the sending end. The curves in Fig. 14 show graphically the variation of the voltage with distance from either end, and the dotted curve shows the voltage at points along the line if the line were open-circuited at those points. See also curves marked E, Figs. 9–13.

39. Variation of Current Values along an Open-Circuited Line.— Equation (50), Chapter IV, gives the current at any point of transmission line in terms of receiving-end quantities:

$$I = I_r \cosh pS_r + \frac{E_r}{Z_0} \sinh pS_r. \tag{39}$$

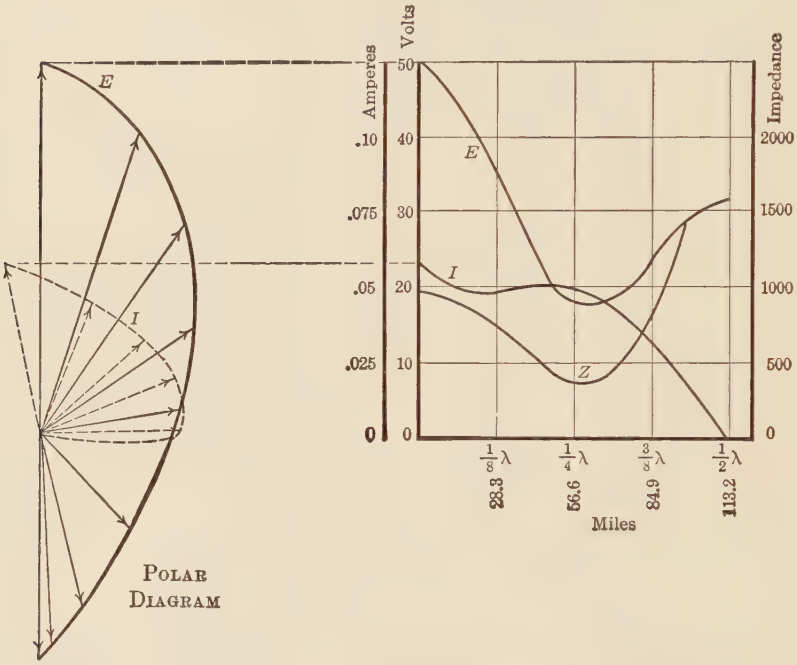

Fig. 11. — Variation of E, I, and Z with Distance. Open-circuited No. 10 B. & S. Line. S = 113.2 Miles

For an open-circuited line $I_r = 0$, hence for an open line (39) becomes

$$I = \frac{E_r}{Z_0} \sinh pS_r = \frac{E_r}{Z_0} \sinh (a + jv)\, S_r. \tag{40}$$

Substituting exponential forms for E_r from (10), and also for the hyperbolic sine of the complex angle, we obtain

$$I = \frac{E_r \epsilon^{-j\beta_1}}{Z_0 \epsilon^{j\alpha}} \sqrt{\cosh^2 aS_r - \cos^2 vS_r}\; \epsilon^{j\beta_4}, \tag{41}$$

where

$$\beta_4 = \tan^{-1} \frac{\tan vS_r}{\tanh aS_r}.$$

When $S_r = 0$, $\cosh aS_r = 1$ and $\cos vS_r = 1$, so that the current is zero at the open end of the line. This was to be expected from the nature of the problem. As we proceed, however, from the

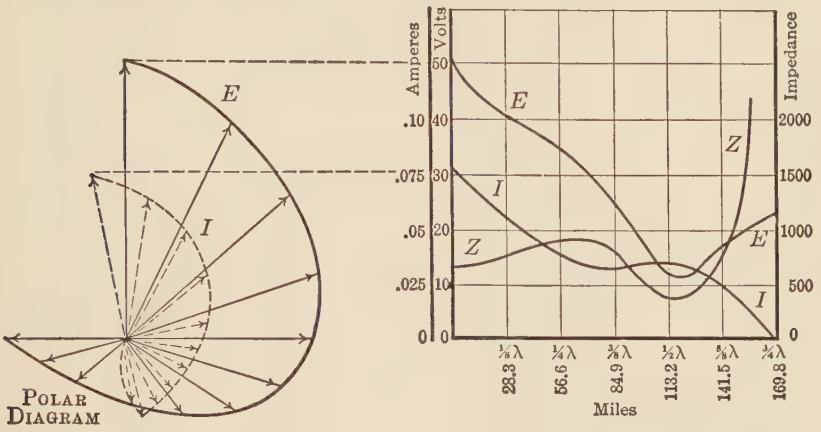

Fig. 12. — Variation of E, I, and Z with Distance. Open-circuited No. 10 B. & S. Line. $S = 169.8$ Miles

open end of the line toward the sending end, the hyperbolic term of the equation increases above the value of 1 and the circular trigonometric term varies periodically between unity and zero.

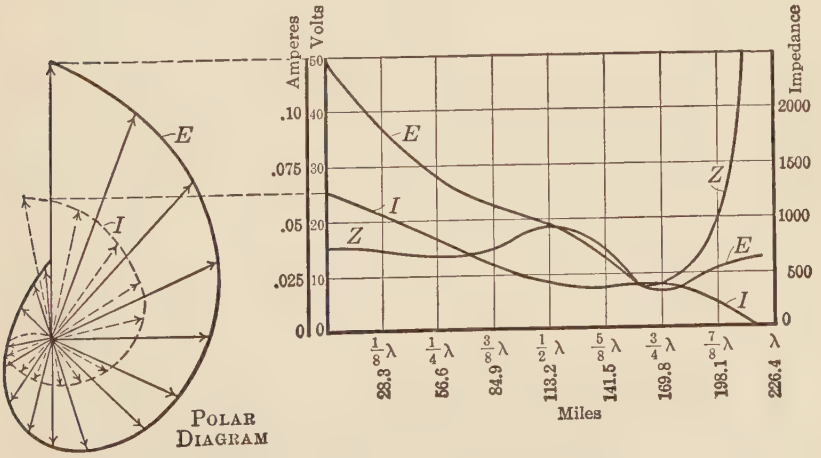

Fig. 13. — Variation of E, I, and Z, with Distance. Open-circuited No. 10 B. & S. Line. $S = 226.4$ Miles

Hence at points along the line where $\cos^2 vS_r = 0$, *i.e.*, where $vS_r = (2n - 1)\pi/2$, the current passes through a maximum value, and at points where $\cos vS_r = 1$, *i.e.*, where $vS_r = 2n\pi/2$, the current

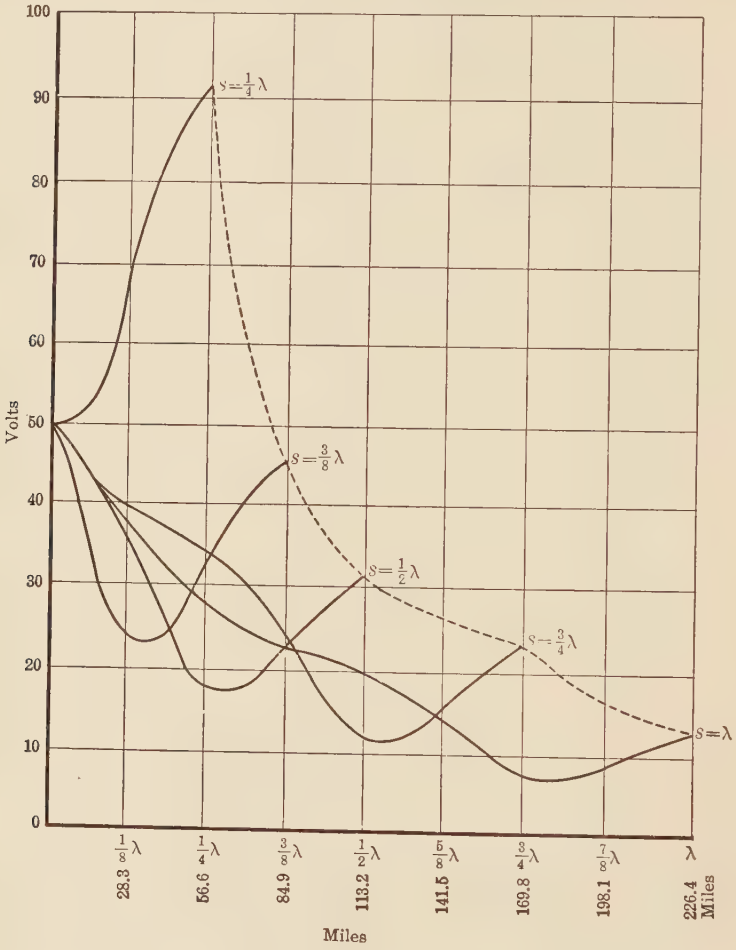

Fig. 14. — Variations of Voltage along Open-circuited No. 10 B. & S. Lines of Different Lengths

will be a minimum. It follows, therefore, that with distances measured from the open end of the line, the current will pass through maxima at every odd quarter wave-length and through minima at distances from that end equal to even quarter wave-lengths.

The curves marked I in Figs. 9 to 13 and the curves in Fig. 15 are obtained by plotting the calculated current values for the No. 10 B. & S. line of different lengths, as given in Table 4, page 68.

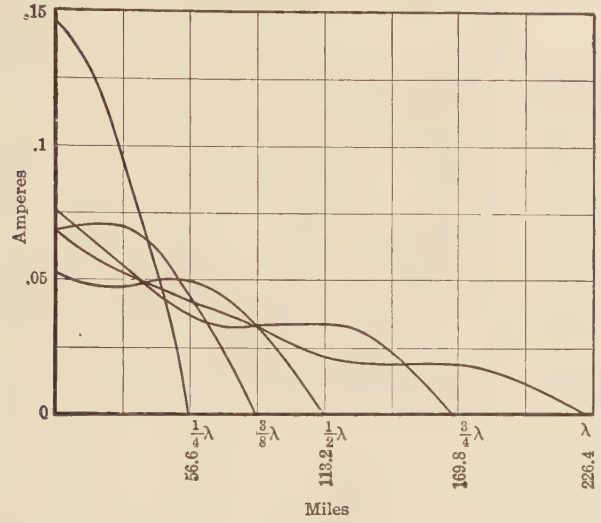

FIG. 15. — Variations of Current along Open-circuited No. 10 B. & S. Lines of Different Lengths

They show the variation of current with distance from either end of the open-circuited line.

40. Variation of Impedance along an Open-Circuited Line. — The impedance at any point along a transmission line is the ratio of the voltage to the current at the point considered. By equations (37) and (40), its value is

$$Z = \frac{E}{I} = \frac{E_r \cosh pS_r}{\frac{E_r}{Z_0} \sinh pS_r},$$

$$Z = \frac{Z_0}{\tanh pS_r} = Z_0 \frac{\cosh (a + jv) S_r}{\sinh (a + jv) S_r}. \tag{42}$$

Using the exponential forms for the characteristic impedance and for the hyperbolic functions of the complex angle, (42) becomes

$$Z = Z_0 \epsilon^{j\alpha} \frac{\sqrt{\cosh^2 aS_r - \sin^2 vS_r}\, \epsilon^{j\beta_3}}{\sqrt{\cosh^2 \sigma S_r - \cos^2 vS_r}\, \epsilon^{j\beta_4}}, \tag{43}$$

or

$$Z = Z_0 \sqrt{\frac{\cosh^2 aS_r - \sin^2 vS_r}{\cosh^2 aS_r - \cos^2 vS_r}} \epsilon^{j\theta}, \tag{44}$$

where

$$\theta = \beta_3 - \beta_4 + \alpha, \tag{45}$$

$$\beta_3 = \tan^{-1} (\tanh aS_r \tan vS_r), \tag{45a}$$

$$\beta_4 = \tan^{-1} \frac{\tan vS_r}{\tanh aS_r}. \tag{45b}$$

Considering the variation of the numerical value of the impedance as given by equation (44), it will be noted that at the receiving end of the line, where $S_r = 0$, $\cosh^2 aS_r = 1$, $\sin^2 vS_r = 0$, and $\cos^2 vS_r = 1$, so that $Z = \infty$, a result that was to be expected, since the line is assumed to be open-circuited. As we proceed toward the sending end, the hyperbolic terms in the equation increase; $\sin^2 vS_r$ and $\cos^2 vS$ vary periodically between zero and unity. It follows that for lines with equal transmission constants but of different lengths, the impedance will be the same for all points equally distant from the receiving end, no matter what the length of the line may be. This effect is shown in Fig. 16 for various lengths of No. 10 B. & S. lines. Proceeding from the open end of the line, since $\sin^2 vS_r$ increases and $\cos^2 vS_r$ decreases with distance, the first reaching a value of 1, and the second a value of 0 when S_r is one-fourth of a wave-length, the impedance decreases as we proceed from the open end of the line, reaching a minimum at a distance from the open end equal to one-quarter of a wave-length. Moreover, since these values of the sine and cosine repeat at every odd quarter wave-length, the impedance will also pass through minima at every quarter wave-length distance and through maxima at distances equal to even quarter wave-lengths measured from the open end of the line. Since the trigonometric terms become more and more insignificant in comparison with the hyperbolic term, as S_r increases, it follows that the above mentioned maxima and minima become less and less apparent as we approach the sending end of the line. The impedance at the sending end will be a minimum for lengths of line equal to an odd number of quarter wave-lengths, which is also evident from the fact that lines of such lengths are in a resonant condition.

Another interesting fact revealed by equation (44) is that at every odd eighth of a wave-length measured from the open end of

the line, the impedance is numerically equal to the characteristic line impedance, for at those points $\cos vS_r = \sin vS_r$ and the radical in that equation is equal to 1. The phase angle of the impedance at these points, however, is not equal to the characteristic phase

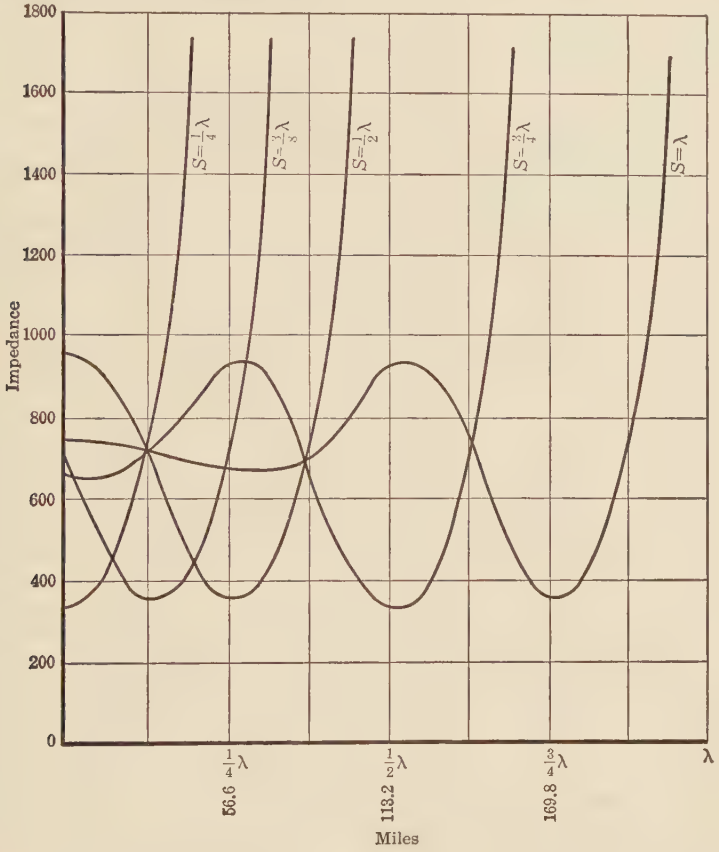

Fig. 16. — Variation of Impedance along Open-circuited No. 10 B. & S. Lines of Different Lengths

angle of the line. Hence the dissipative and reactive components are not equal to the respective components of the characteristic line impedance.

Table 4, page 68, gives the calculated values of the impedance for the No. 10 B. & S. open-circuited line for lengths of line as indicated, and the curves marked Z, Figs. 9 to 13, inclusive, and also the

TABLE 4

S_r	$\frac{S_r}{\lambda}$	S = 56.6 miles = $\frac{1}{4}\lambda$.			S = 84.8 miles = $\frac{3}{8}\lambda$.			S = 113.2 = $\frac{1}{2}\lambda$.		
		E	I	Z	E	I	Z	E	I	Z
0	0	$92.7\ \epsilon^{-j90°}$	$0\ \epsilon^{j7.1°}$	$\infty\ \epsilon^{j82.9°}$	$45.5\ \epsilon^{-j147.3°}$	$0\ \epsilon^{-j61.1°}$	$\infty\ \epsilon^{-j82.9°}$	$31.6\ \epsilon^{-j180°}$	$0\ \epsilon^{j97.1°}$	$\infty\ \epsilon^{-j82.9°}$
14.1	$\frac{1}{16}$	$86\ \epsilon^{-j87°}$	$.0513\ \epsilon^{-j6.2°}$	$1671\ \epsilon^{-j80.8°}$	$42.3\ \epsilon^{-j144.3°}$	$.0253\ \epsilon^{-j64.5°}$	$1671\ \epsilon^{-j80.8°}$	$29.4\ \epsilon^{-j177°}$	$.0176\ \epsilon^{j96.2°}$	$1671\ \epsilon^{-j80.8°}$
28.2	$\frac{1}{8}$	$68\ \epsilon^{-j75.8°}$	$.0942\ \epsilon^{j3.1°}$	$723\ \epsilon^{j72.7°}$	$33.4\ \epsilon^{-j133.1°}$	$.0463\ \epsilon^{-j60.1°}$	$723\ \epsilon^{-j72.7°}$	$23.2\ \epsilon^{-j165.8°}$	$.0321\ \epsilon^{-j93.1°}$	$723\ \epsilon^{j72.7°}$
42.3	$\frac{3}{16}$	$51.7\ \epsilon^{-j48.4°}$	$.128\ \epsilon^{j24.4°}$	$404\ \epsilon^{j50.8°}$	$25.3\ \epsilon^{-j105.7°}$	$.0627\ \epsilon^{-j54.9°}$	$404\ \epsilon^{-j50.8°}$	$17.6\ \epsilon^{-j138.4°}$	$.0437\ \epsilon^{-j84.1°}$	$404\ \epsilon^{-j50.8°}$
56.6	$\frac{1}{4}$	$50\ \epsilon^{j0°}$	$.145\ \epsilon^{+j11.1°}$	$344\ \epsilon^{-j11.1°}$	$24.4\ \epsilon^{-j57.3°}$	$.0710\ \epsilon^{-j46.2°}$	$344\ \epsilon^{-j11.1°}$	$17.0\ \epsilon^{-j90°}$	$.0495\ \epsilon^{-j78.9°}$	$344\ \epsilon^{-j11.1°}$
70.7	$\frac{5}{16}$	.....	.....	.....	$36.2\ \epsilon^{-j21.2°}$	$.0704\ \epsilon^{-j32.9°}$	$513\ \epsilon^{+j11.7°}$	$25.1\ \epsilon^{-j54.9°}$	$.0490\ \epsilon^{-j66.6°}$	$513\ \epsilon^{+j11.7°}$
84.8	$\frac{3}{8}$	.....	.....	.....	$50\ \epsilon^{-j0°}$	$.0697\ \epsilon^{-j13.4°}$	$728\ \epsilon^{j13.4°}$	$34.8\ \epsilon^{-j32.7°}$	$.0477\ \epsilon^{-j46.1°}$	$728\ \epsilon^{+j13.4°}$
98.9	$\frac{7}{16}$	.....	.....	.....	.....	.....	.....	$43.3\ \epsilon^{-j17.5°}$	$.0476\ \epsilon^{-j18.9°}$	$910\ \epsilon^{j1.4°}$
113.2	$\frac{1}{2}$	.....	.....	.....	.....	.....	.....	$50\ \epsilon^{-j0°}$	$.0536\ \epsilon^{+j11.1°}$	$932\ \epsilon^{-j11.1°}$

S_r	$\frac{S_r}{\lambda}$	S = 169.8 miles = $\frac{3}{4}\lambda$.			S = 226.4 miles = λ.		
		E	I	Z	E	I	Z
0	0	$22.1\ \epsilon^{-j270.9°}$	$0\ \epsilon^{-j187.1°}$	$\infty\ \epsilon^{-j82.9°}$	$12.5\ \epsilon^{-j360°}$	$0\ \epsilon^{-j277.1°}$	$\infty\ \epsilon^{-j82.9°}$
14.1	$\frac{1}{16}$	$20.5\ \epsilon^{-j267°}$	$.0135\ \epsilon^{-j186.0°}$	$1671\ \epsilon^{-j80.8°}$	$11.6\ \epsilon^{-j357°}$	$.00695\ \epsilon^{-j276.2°}$	$1671\ \epsilon^{-j80.6°}$
28.2	$\frac{1}{8}$	$16.2\ \epsilon^{-j255.8°}$	$.0224\ \epsilon^{-j184.1°}$	$723\ \epsilon^{-j72.7°}$	$9.2\ \epsilon^{-j345.8°}$	$.0127\ \epsilon^{-j273.1°}$	$723\ \epsilon^{-j72.7°}$
42.3	$\frac{3}{16}$	$12.3\ \epsilon^{-j992.4°}$	$.0305\ \epsilon^{-j177.6°}$	$404\ \epsilon^{-j50.8°}$	$7.1\ \epsilon^{-j318.4°}$	$.0175\ \epsilon^{-j267.6°}$	$404\ \epsilon^{-j50.8°}$
56.6	$\frac{1}{4}$	$11.9\ \epsilon^{-j180°}$	$.0346\ \epsilon^{-j168.9°}$	$344\ \epsilon^{-j11.1°}$	$6.72\ \epsilon^{j270°}$	$.01955\ \epsilon^{-j258.9°}$	$344\ \epsilon^{-j11.1°}$
70.7	$\frac{5}{16}$	$17.5\ \epsilon^{-j143.7°}$	$.0311\ \epsilon^{-j155.7°}$	$513\ \epsilon^{+j11.7°}$	$9.9\ \epsilon^{-j203.9°}$	$.0193\ \epsilon^{-j245.6°}$	$513\ \epsilon^{+j11.7°}$
84.8	$\frac{3}{8}$	$22.3\ \epsilon^{j122.7°}$	$.0306\ \epsilon^{-j136.1°}$	$728\ \epsilon^{+j13.4°}$	$13.73\ \epsilon^{-j242.7°}$	$.0187\ \epsilon^{-j226.1°}$	$723\ \epsilon^{+j13.4°}$
98.8	$\frac{7}{16}$	$30.3\ \epsilon^{-j107.5°}$	$.0333\ \epsilon^{-j108.9°}$	$910\ \epsilon^{+j1.4°}$	$17.15\ \epsilon^{-j197.5°}$	$.0188\ \epsilon^{-j188.9°}$	$910\ \epsilon^{+j1.4°}$
113.2	$\frac{1}{2}$	$34.9\ \epsilon^{-j90°}$	$.0374\ \epsilon^{-j78.9°}$	$932\ \epsilon^{-j11.1°}$	$19.75\ \epsilon^{-j180°}$	$.0212\ \epsilon^{-j168.9°}$	$932\ \epsilon^{-j11.1°}$
127.3	$\frac{9}{16}$	$37.2\ \epsilon^{j-71.2°}$	$.0457\ \epsilon^{-j82.2°}$	$815\ \epsilon^{-j19°}$	$21\ \epsilon^{-j161.2°}$	$.0258\ \epsilon^{-j142.2°}$	$815\ \epsilon^{-j19°}$
141.4	$\frac{5}{8}$	$40.3\ \epsilon^{-j49.4°}$	$.0558\ \epsilon^{-j29.6°}$	$723\ \epsilon^{-j19.8°}$	$22.7\ \epsilon^{-j1.94°}$	$.0314\ \epsilon^{-j119.6°}$	$723\ \epsilon^{-j19.8°}$
155.5	$\frac{11}{16}$	$43.5\ \epsilon^{-j25°}$	$.0652\ \epsilon^{-j9.1°}$	$667\ \epsilon^{-j15.9°}$	$24.6\ \epsilon^{-j115°}$	$.0369\ \epsilon^{-j99.1°}$	$667\ \epsilon^{-j15.9°}$
169.8	$\frac{3}{4}$	$50\ \epsilon^{j0°}$	$.0750\ \epsilon^{+j11.1°}$	$667\ \epsilon^{-j11.1°}$	$28.2\ \epsilon^{-j90°}$	$.0423\ \epsilon^{-j78.9°}$	$667\ \epsilon^{-j11.1°}$
183.9	$\frac{13}{16}$	.....	.....	.....	$32.6\ \epsilon^{-j67°}$	$.0469\ \epsilon^{-j67.8°}$	$695\ \epsilon^{-j9.2°}$
198	$\frac{7}{8}$	.....	.....	.....	$38\ \epsilon^{-j43.4°}$	$.0526\ \epsilon^{-j55.5°}$	$723\ \epsilon^{-j7.9°}$
212.1	$\frac{15}{16}$	.....	.....	.....	$43.7\ \epsilon^{-j21.4°}$	$.0586\ \epsilon^{-j12.3°}$	$746\ \epsilon^{-j9.1°}$
226.4	1	.....	.....	.....	$50\ \epsilon^{-j0°}$	$.067\ \epsilon^{+j11.1°}$	$744\ \epsilon^{-j11.1°}$

In entries with modulus 0 or ∞, the argument represents a limiting value.

curves Fig. 16, illustrate the variation of impedance with distance from either end of the line, as discussed above.

41. Variation of the Phase Angle between Voltage and Current along an Open-Circuited Line. — Equation (45)

$$\theta = \beta_3 - \beta_4 + \alpha$$

gives the phase angle between the voltage and the current at any point along an open-circuited line. The angle α, which is the characteristic phase angle, depends upon the electrical constant of the line and upon the frequency of the supply, while β_3 and β_4 depend upon the transmission constants a and v and upon the length of the line, as shown by the two equations

$$\beta_3 = \tan^{-1} (\tanh aS_r \tan vS_r), \tag{46}$$

$$\beta_4 = \tan^{-1} \frac{\tan vS_r}{\tanh aS_r}. \tag{47}$$

To obtain the law of variation of the phase angle θ with distance along the line, equation (45) may be written in the form

$$\tan \theta = \tan [(\beta_3 - \beta_4) + \alpha] = \frac{\tan (\beta_3 - \beta_4) + \tan \alpha}{1 - \tan \alpha \tan (\beta_3 - \beta_4)}. \tag{48}$$

But we have

$$\tan (\beta_3 - \beta_4) = \frac{\tan \beta_3 - \tan \beta_4}{1 + \tan \beta_3 \tan \beta_4}.$$

Substituting for $\tan \beta_3$ and $\tan \beta_4$ their values from (46) and (47), respectively, we get

$$\tan (\beta_3 - \beta_4) = \frac{\tanh aS_r \tan vS_r - \dfrac{\tan vS_r}{\tanh aS_r}}{1 + \tanh aS_r \tan vS_r \dfrac{\tan vS_r}{\tanh aS_r}}$$

$$= \frac{(\tanh^2 aS_r - 1) \tan vS_r}{(\tan^2 vS_r + 1) \tanh aS_r},$$

or

$$\tan (\beta_3 - \beta_4) = \frac{(\sinh^2 aS_r - \cosh^2 aS_r) \sin vS_r \cos vS_r}{(\sin^2 vS_r + \cos^2 vS_r) \sinh aS_r \cosh aS_r}$$

$$= \frac{-\sin 2vS_r}{\sinh 2aS_r}. \tag{49}$$

Substituting this in equation (48), we obtain

$$\tan\theta = \frac{\dfrac{-\sin 2vS_r}{\sinh 2aS_r} + \tan\alpha}{1 + \tan\alpha\dfrac{\sin 2vS_r}{\sinh 2aS_r}},$$

or

$$\tan\theta = \frac{-\sin 2vS_r + \tan\alpha \sinh 2aS_r}{\sinh 2aS_r + \tan\alpha \sin 2vS_r}. \tag{50}$$

While it might be easier to compute the phase angle θ for any point along the open line by calculating the angles β_3 and β_4 and

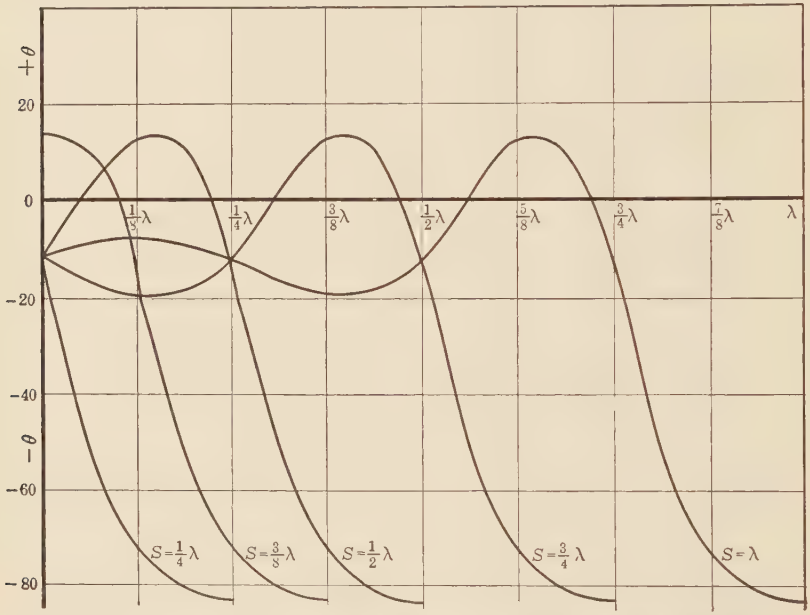

Fig. 17. — Variation of Phase Angle with Distance. Open-circuited No. 10 B. & S. Lines of Different Lengths

using equation (45), the above equation will be found very convenient in the investigation of the phase angle at special points, as will be seen in the following two sections.

Figure 17 shows the variation of the phase angle θ with distance for the illustrative No. 10 B. & S. line of different lengths. It is obtained by subtracting the arguments of the current vector from the arguments of the voltage vector as given in Table 4, page 68.

42. Phase Angle at the Open End of a Line. — At the open end of a line, the current is zero, and the voltage has a definite value depending upon the length of the line relative to the wavelength. Since the energy at that point is all stored in the electrostatic field, it would seem the phase angle would approach a 90° leading value as we approach the open end. While it is true that the angle is leading, it cannot be 90° unless the line has no leakage. This may be shown as follows: For very small angles (circular and hyperbolic), the sine of the angle is very nearly equal to the angle itself expressed in radians (circular or hyperbolic). As we approach the open end, *i.e.*, as S_r approaches zero, we have approximately

$$\sin 2\,vS_r = 2\,vS_r, \quad \text{and} \quad \sinh 2\,aS_r = 2\,aS_r.$$

The phase angle near the open end of the line, as expressed by equation (50), may be written in the form

$$\tan\theta = \frac{-2\,vS_r + 2\,aS_r \tan\alpha}{2\,aS_r + 2\,vS_r \tan\alpha}.$$

Dividing by $2\,S_r$, which is a common factor, we get

$$\tan\theta = \frac{-v + a\tan\alpha}{a + v\tan\alpha}. \tag{51}$$

This expression indicates that the phase angle at the open-circuited end of the line is independent of the length of the line.

If we substitute, however, for v and a in terms of the propagation constant p, as given by equations (44) and (45), Chapter IV, equation (51) becomes

$$\tan\theta = \frac{-p\sin\delta + p\cos\delta\tan\alpha}{p\cos\delta + p\sin\delta\tan\alpha},$$

and since p cancels, we get

$$\tan\theta = \frac{-\sin\delta + \cos\delta\dfrac{\sin\alpha}{\cos\alpha}}{\cos\delta + \sin\delta\dfrac{\sin\alpha}{\cos\alpha}}$$

$$= \frac{-\sin\delta\cos\alpha + \cos\delta\sin\alpha}{\cos\delta\cos\alpha + \sin\delta\sin\alpha}$$

$$= \frac{-\sin(\delta-\alpha)}{\cos(\delta-\alpha)} = -\tan(\delta-\alpha).$$

By equation (43*a*) Chapter IV, we have

$$\delta = \frac{\alpha_1 + \alpha_2}{2},$$

and by equation (58) Chapter IV,

$$\alpha = \frac{\alpha_1 - \alpha_2}{2}.$$

Hence

$$\delta - \alpha = \frac{\alpha_1 + \alpha_2}{2} - \frac{\alpha_1 - \alpha_2}{2} = \alpha_2,$$

or

$$\theta = -\alpha_2, \tag{52}$$

and since

$$\alpha_2 = \tan^{-1}\frac{C\omega}{G},$$

it follows that the phase angle at the open end of the line depends only upon the capacitance and the leakage conductance of the line and upon the frequency, and can be 90° only when the leakage conductance $G = 0$.

43. Points along an Open-Circuited Line at which Voltage and Current are in Phase. — When the voltage and current are in phase with one another at any one point along the line, the angle $\theta = 0$. When this happens equation (50) becomes

$$\tan \alpha = \frac{\sin 2\,vS_r}{\sinh 2\,aS_r}. \tag{53}$$

This equation cannot be solved directly for S_r, since S_r appears in terms of a hyperbolic and a circular trigonometric function. However, the value of S_r, *i.e.*, the distance from the receiving end at which $\theta = 0$, may be found graphically, by plotting to the same scale the quantities

$$\frac{\sin 2\,vS_r}{\tan \alpha} \text{ against } S_r,$$

and

$$\sinh 2\,aS_r \text{ against } S_r.$$

The values of S_r at which the two curves intersect are the distances from the receiving end at which the voltage and current are in phase with one another.

Since $\tan \alpha$ is a constant and generally negative, the first curve is generally a reversed sine curve of double frequency, the length

of line determining the number of complete alternations. Thus, if the line under consideration is equal in length to one complete wave, the curve for sin $2vS_r/\tan\alpha$ against S_r will have four complete alternations. If the line is equal to one-quarter wave-length, the curve will be one-half of a sine curve. The graph

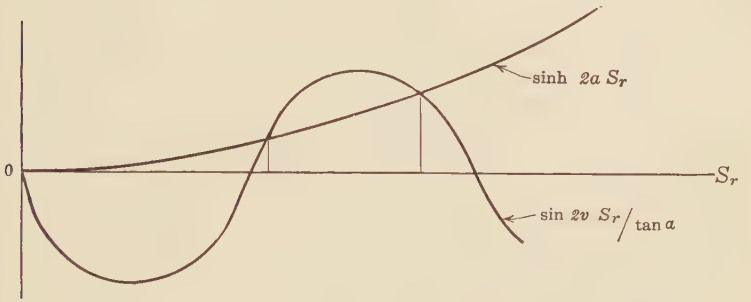

FIG. 18

in Fig. 18 is for a line somewhat longer than one-half of a wave-length.

The relationship between these two curves, as shown in the figure, indicates that the number of points along an open-circuited line at which the voltage and current are in phase, depends upon the relative values of the constants of and the length of the line. Thus for a line with zero attenuation, *i.e.*, for a hypothetical line with no resistance and no leakage, the curve for sinh 2 aS_r against S_r would coincide with the horizontal axis, since sinh 2 $aS_r = 0$ for all values of length of line. For such a line, the current and the voltage are in phase at points distant from one another by one-quarter of a wave-length measured from the open end of the line. It follows that the maximum possible number of points at which the voltage and current are in phase may be determined by the length of the line in terms of the wave-length. Thus, under no condition will the phase angle be zero for open-circuited lines shorter than one-quarter of a wave-length. A line whose length is less than one-half of a wave-length cannot have more than two points at which the voltage is in phase with the current, and so on.

The fact that practical lines have both resistance and leakage conductance modifies the above relationship, depending upon the relative values of the attenuation constant and the wave-length constant. A practical line must be longer than one-quarter of a

wave-length to have one point at which $\theta = 0$, and it may be shorter than one-half of a wave-length and still have two points at which the current is in phase with the voltage.

The following table gives the calculated data and Fig. 19 gives the graph from which the points of zero phase angle were obtained for the illustrative No. 10 B. & S. open-circuited line.

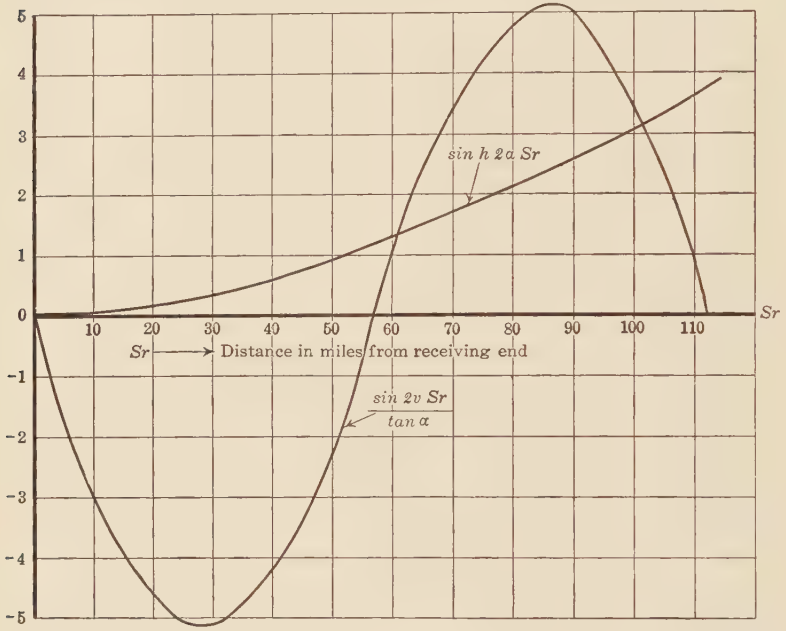

Fig. 19

The graph shows that for this particular line there cannot be more than two points at which the current and the voltage are in phase with each other. The first point is 62.5 miles, and the second approximately 100 miles, from the open-circuited end of the line. No matter what the line length might be, these points will always occur at those distances from the open end of the line. In other words, the line must be at least 62.5 miles long to have one point of zero phase angle, which in that case will be at the sending end of the line, making the sending-end current in phase with the sending-end voltage. The line must be at least 100 miles long to have two points of zero phase angle.

TABLE 5

S_r	$\frac{\sin 2\,vS_r}{\tan \alpha}$	$\sinh 2\,aS_r$
0	0	0
14.1	− 3.575	.2608
28.2	− 5.12	.5381
42.3	− 3.575	.8523
56.6	0	1.2251
70.7	+ 3.575	1.6788
84.8	+ 5.12	2.2447
98.9	+ 3.575	2.9609
113.2	0	3.8753

For the stretch of line between the open end and the first point at which $\theta = 0$, the current leads the voltage by an angle which is gradually decreasing to zero at the first point from the end of the line; and for the part of the line between the two points at which $\theta = 0$, the current lags behind the voltage.

It will be shown in Chapter IX that, at the point where $\theta = 0$, the energy transferred has no reactive component, *i.e.*, there is no energy stored either in the electrostatic or in the magnetic state.

44. Points along an Open-Circuited Line at which the Phase Angle is Equal to the Characteristic Phase Angle. — The condition assumed here is that $\theta = \alpha$, whence equation (25) becomes

$$\alpha = \beta_3 - \beta_4 + \alpha, \quad \text{or} \quad \beta_3 - \beta_4 = 0,$$

so that

$$\tan (\beta_3 - \beta_4) = 0.$$

Since we have, by (49),

$$\tan (\beta_3 - \beta_4) = \frac{-\sin 2\,vS_r}{\sinh 2\,aS_r},$$

it follows that

$$\frac{-\sin 2\,vS_r}{\sinh 2\,aS_r} = 0,$$

or $\sin 2\,vS_r = 0$, which is true when $2\,vS_r = n\pi$, where n is any integer. Solving the above equation for S_r, we get

$$S_r = n\,\frac{\pi}{2\,v},$$

and since $v = 2\pi/\lambda$, it follows that

$$S_r = n\frac{\lambda}{4}.$$

This result shows that the voltage and the current will differ in phase by an angle equal to the characteristic phase at every quarter wave length measured from the open-circuited end of the line. The curves in Fig. 17 show the variation of the phase angle with distance for several lengths of No. 10 B. & S. open-circuited lines, and they illustrate fully the results obtained in the last four sections of this chapter.

CHAPTER VII

THE SHORT-CIRCUITED LINE

The detailed discussion of the open-circuited line given in the preceding chapter will enable us to abbreviate that of the short-circuited line, for the methods of analysis, keeping in mind the difference in line condition, are quite similar.

45. Sending-end Impedance of the Short-Circuited Line. — The assumption that the line is short-circuited at its far end, implies that the voltage at that end is zero. Whence, if the line under consideration is of a length equal to S miles, the general transmission equation for voltage for the condition $E_r = 0$, is

$$E_s \cosh pS - I_s Z_0 \sinh pS = 0, \tag{1}$$

whence

$$Z_0 = \frac{E_s}{I_s} \frac{\cosh pS}{\sinh pS} = \frac{Z_s}{\tanh pS}. \tag{2}$$

Solving for Z_s, we get

$$Z_s = Z_0 \tanh pS. \tag{3}$$

To calculate the value of Z_s and to determine the law of variation of the sending-end impedance with line-length, the above expression must be converted into the more convenient form

$$Z_s = Z_0 \epsilon^{j\alpha} \sqrt{\frac{\cosh^2 aS - \cos^2 vS}{\cosh^2 aS - \sin^2 vS}}\, \epsilon^{j(\beta_2 - \beta_1)}, \tag{3a}$$

where

$$\beta_1 = \tan^{-1} (\tanh aS \tan vS),$$

and

$$\beta_2 = \tan^{-1} \frac{\tan vS}{\tanh aS}.$$

An analysis of the above equation for the sending-end impedance shows that starting with an impedance value equal to zero when $S = 0$, *i.e.*, when the line is short-circuited at the sending-end, the sending-end impedance increases, passing through maxima for lines whose lengths are odd quarter wave-lengths, and through minima for lines whose lengths are even quarter wave-lengths.

It might be of interest to note, that since $\cos vS = \sin vS$ for lengths of line equal to one-eighth of a wave-length, or an odd

multiple of that length, the sending-end impedance for such lengths of line will be numerically equal to the characteristic line impedance. The phase angles, however, are not equal. Hence the reactive and energy-consuming components, respectively, are not equal to those of the characteristic line impedance.

46. Sending-End Current. — By equation (3), the sending-end current of a short-circuited line of length S, is

$$(4) \qquad I_s = \frac{E_s}{Z_0 \tanh pS} = \frac{E_s \cosh pS}{Z_0 \sinh pS},$$

which, for numerical calculations, becomes

$$(5) \qquad I_s = \frac{E_s}{Z_0} \sqrt{\frac{\cosh^2 aS - \sin^2 vS}{\cosh^2 aS - \cos^2 vS}}\, \epsilon^{j(\beta_1 - \beta_2 - \alpha)}$$

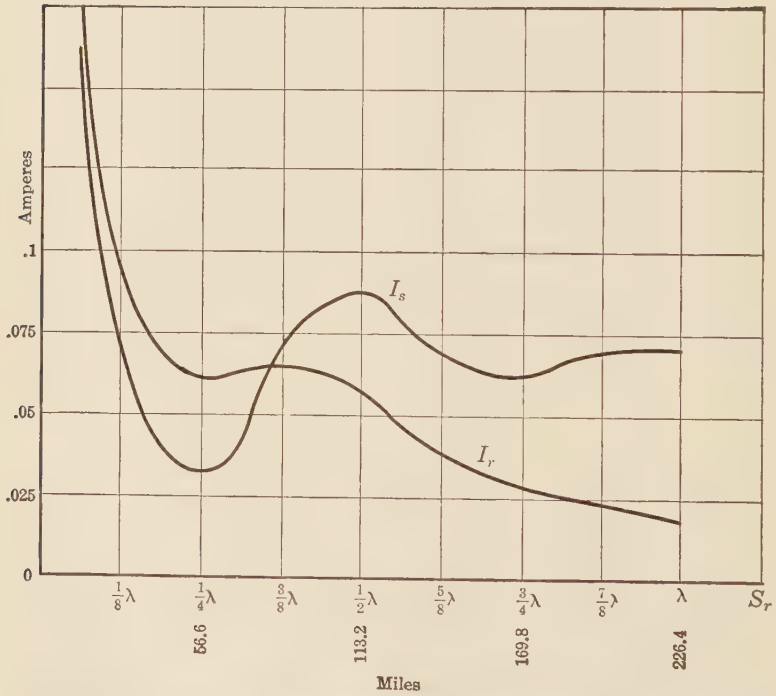

FIG. 20. — Variation of Sending-end and Receiving-end Currents with Length of Short-circuited Lines

An analysis of this expression shows that starting with a current value equal to infinity, if the line were short-circuited at the send-

ing end, the sending-end current will be a minimum when the length of the line is one-quarter of a wave-length long, or an odd multiple of that length, and a maximum when the line is one-half of a wave-length, or any even number of quarter wave-lengths. Somewhat paradoxically then, there is less danger from a short-circuit at the end of a line one-quarter of a wave-length long, than from a short-circuit at the end of line twice that length. The curve marked I_s, Fig. 20, shows the variation of the sending-end current of short-circuited No. 10 B. & S. lines, with the line-length.

47. Current at the Short-Circuited End. — The current at the end of any transmission line is

$$I_r = I_s \cosh pS - \frac{E_s}{Z_0} \sinh pS.$$

Substituting for I_s its value from (4), we get

$$(6) \qquad I_r = \frac{E_s}{Z_0} \frac{\cosh^2 pS - \sinh^2 pS}{\sinh pS} = \frac{E_s}{Z_0 \sinh pS},$$

which, for numerical calculations, becomes

$$(7) \qquad I_r = \frac{E_s}{Z_0 \sqrt{\cosh^2 aS - \cos^2 vS}} \epsilon^{-j(\alpha + \beta_2)}.$$

This equation shows that starting with a value equal to infinity if the line were short-circuited at the sending end ($S = 0$), the current at the short-circuited end decreases as the line-length increases, passing through alternate minima and maxima as the line is an odd or an even number of quarter wave-lengths, respectively. Both minima and maxima become less and less apparent as the line is made very long. The curve marked I_r, Fig. 20, gives the variation of the current at the short-circuited end of No. 10 B. & S. lines with the line-length.

It is of interest to note from an examination of the relative values of the sending-end current and the short-circuited-end current, as shown in Fig. 20, that up to a certain definite length of line, the current at the short-circuited end of the line is larger than the current at the sending end. This is analogous to the Ferranti effect on open circuit, namely, that voltage at the open end of a line may be larger than the sending voltage. The conditions under which this phenomenon will occur may be shown to be the same as for the Ferranti effect, namely, that the attenuation constant must be less than 56.09 per cent of the wave-length

constant. Moreover, this phenomenon will occur only between the limits of line-length for which the Ferranti effect will occur, namely, between the limits of $S = 0$, where both sending-end current and short-circuited-end current are the same and approaching infinity, and the line-length for which $I_s = I_r$. By (5) and (7), this length is seen to be that given by the equation

$$\frac{\cosh^2 aS - \sin^2 vS}{\cosh^2 aS - \cos^2 vS} = \frac{1}{\cosh^2 aS - \cos^2 vS},$$

or

$$\cosh^2 aS - \sin^2 vS = 1,$$

or

$$-\sin^2 vS = 1 - \cosh^2 aS,$$

or

$$-\sin^2 vS = -\sinh^2 aS,$$

or

$$\sin vS = \sinh aS.$$

This result indicates that this effect will occur for the same line-length and between the same limits of line-length as the Ferranti effect, as may be seen also by comparing the curves in Fig. 20 for the point where $I_s = I_r$, with the dotted curve in Fig. 14 for the point where $E_r = E_s$.

48. Voltage at any Point along a Short-Circuited Line. — In terms of receiving-end quantities, the voltage at any point along a transmission line is

$$E = E_r \cosh pS_r + I_r Z_0 \sinh pS_r.$$

Since $E_r = 0$ at the short-circuited end, it follows that

$$E = I_r Z_0 \sinh pS_r. \tag{8}$$

For a definite length of line, the current at the snort-circuited end has the definite value given in equation (7). Using this value of I_r and also exponentials for the terms of equation (8), we obtain

$$E = I_r \epsilon^{-j(\alpha+\beta_2)} Z_0 \epsilon^{j\alpha} \sqrt{\cosh^2 aS_r - \cos^2 vS_r}\, \epsilon^{j\beta_4},$$

or

$$E = I_r Z_0 \sqrt{\cosh^2 aS_r - \cos^2 vS_r}\, \epsilon^{j(\beta_4-\beta_2)}, \tag{9}$$

where

$$\beta_2 = \tan^{-1} \frac{\tan vS}{\tanh aS},$$

and

$$\beta_4 = \tan^{-1} \frac{\tan vS_r}{\tanh aS_r},$$

the angle β_2 being a constant, and the angle β_4 varying with the distance S_r, as measured from the short-circuited end of the line.

Expression (9), by means of which the voltage at any point of a short-circuited line can be calculated, shows that starting with a zero value at the short-circuited end, the voltage, on rising toward the sending end, passes through maxima and minima at every odd and every even quarter wave-length, respectively, as measured from the short-circuited end. These maxima and minima become less and less apparent as we approach the sending end, because the trigonometric term becomes smaller and smaller in comparison with the magnitude of the hyperbolic term under the radical.

Table 6 gives the calculated values of the voltage for No. 10 B. & S. short-circuited lines of lengths as indicated, and at intervals along the respective lines equivalent to one-sixteenth of a wave-length. The curves, marked E in Figs. 21 to 23, show the variation of the voltage as a function of the distance from either end of the line, and corresponding values of current and impedance. The polar diagrams on the left, which give the values of the voltage as a vector with reference to the sending-end voltage, may be used to obtain the instantaneous space values, *i.e.*, the actual wave. Figure 24 shows the variation of the voltage with distance for lengths of No. 10 B. & S. short-circuited lines as indicated, and also how the length of line affects the distribution of potential.

49. Current at any Point along a Short-Circuited Line. — The current at any point along a transmission line in terms of receiving-end quantities is

$$I = I_r \cosh pS_r + \frac{E_r}{Z_0} \sinh pS_r.$$

Since for the line under consideration, $E_r = 0$, it follows that

$$I = I_r \cosh pS_r. \tag{10}$$

Using the exponential form for I_r, and also for the hyperbolic cosine of the complex angle pS_r, we have

$$I = I_r \sqrt{\cosh^2 aS_r - \sin^2 vS_r}\, e^{j(\beta_3 - \beta_2 - \alpha)}. \tag{11}$$

Starting at the short-circuited end with the definite value I_r, the current depending upon the constants of the line and the length of the line, increases toward the sending end, passing through maxima and minima at every even and odd quarter wave-length, respectively, as measured from the short-circuited end of the line.

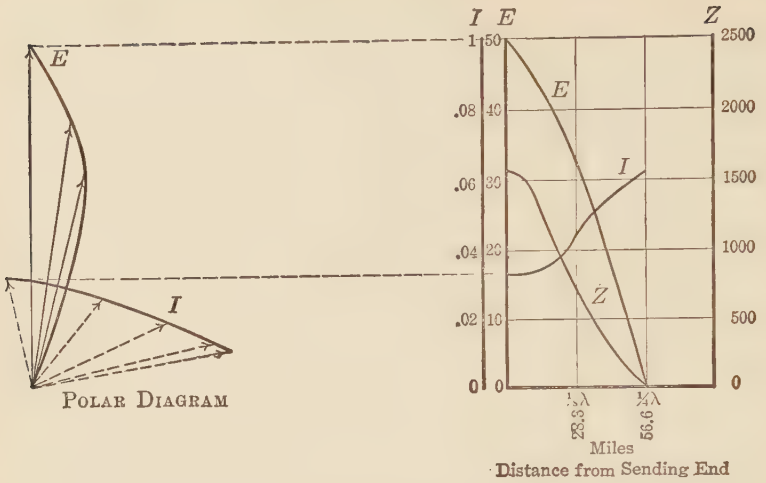

Fig. 21. — Variation of E, I, and Z with Distance. Short-circuited No. 10 B. & S. Line. $S = 56.6$ Miles

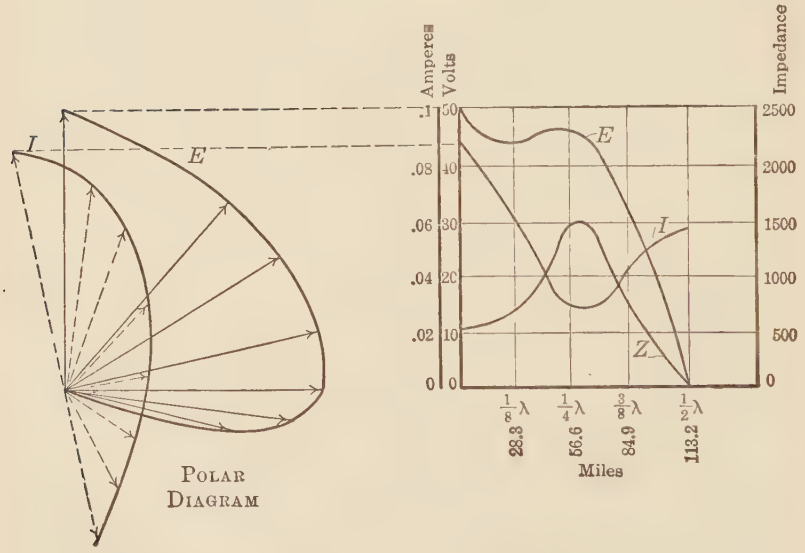

Fig. 22. — Variation of E, I, and Z with Distance. Short-circuited No. 10 B. & S. Line. $S = 113.2$ Miles

The curves marked I in Figs. 21 to 23, inclusive, are obtained by plotting the calculated values of the current for different lengths

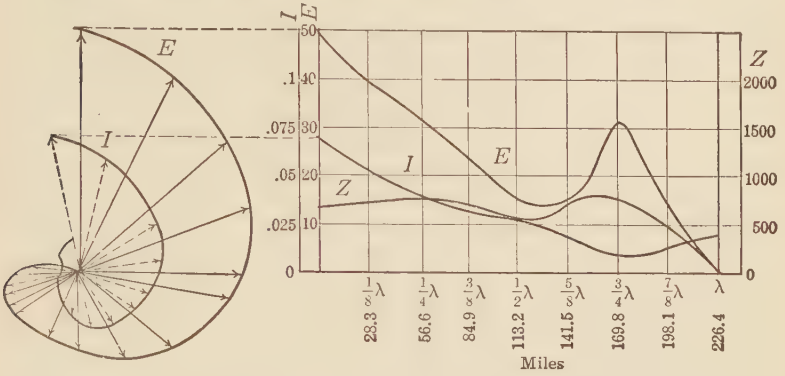

Fig. 23. — Variation of E, I, and Z with Distance. Short-circuited No. 10 B. & S. Line. S = 226.4 Miles

of short-circuited No. 10 B. & S. lines, as given in Table 6. They show the variation of the current as a function of the distance

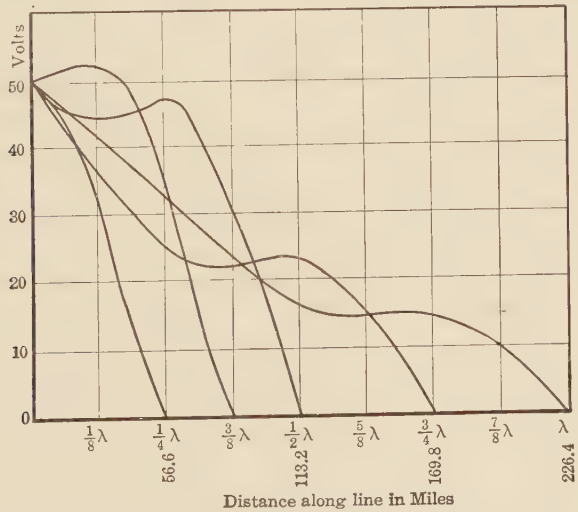

Fig. 24. — Variations of Voltage along No. 10 B. & S. Short-circuited Lines of Different Lengths

from either end of the lines, and also with reference to corresponding values of voltage and impedance. The polar diagrams on the

left of the figures give the variation of the current values as vectors with reference to sending-end voltage, and may be used to obtain the instantaneous space current values, *i.e.*, the actual current wave. It must be kept in mind that since effective values are used, each current value must be multiplied by $\sqrt{2}$, to convert from the effective to the maximum values.

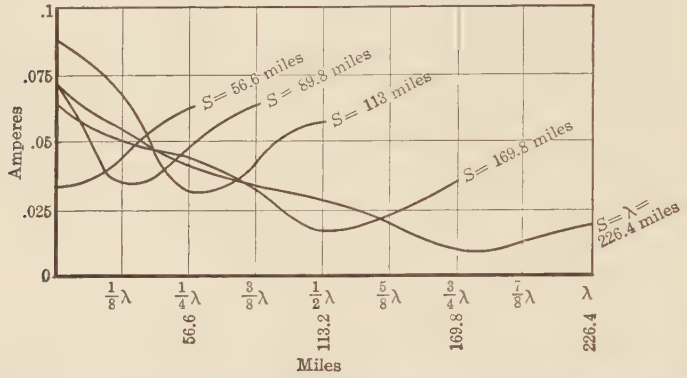

FIG. 25. — Variation of Current along No. 10 B. & S. Short-circuited Lines of Different Lengths.

The curves in Fig. 25 show the relative variation in current with distance as affected by length of line.

50. Impedance at any Point along a Short-Circuited Line. — Since the impedance at any point is the ratio of the voltage to the current at the point under consideration, it follows from (8) and (10) that

$$Z = \frac{I_r Z_0 \sinh pS_r}{I_r \cosh pS_r} = Z_0 \tanh pS_r. \tag{12}$$

Using exponentials, this becomes

$$Z = Z_0 \sqrt{\frac{\cosh^2 aS_r - \cos^2 vS_r}{\cosh^2 aS_r - \sin^2 vS_r}}\, e^{j\theta}, \tag{13}$$

where

$$\theta = \beta_4 - \beta_3 + \alpha \tag{14}$$

is the phase angle between voltage and current at any point. This expression for Z shows that the impedance, starting with a zero value at the short-circuited end of the line, increases toward the sending end, passing periodically through maxima and minima at

TABLE 6

S_r	$S = 56.6$ miles $= \frac{1}{4}\lambda$			$S = 84.8$ miles $= \frac{3}{8}\lambda$			$S = 113.2$ miles $= \frac{1}{2}\lambda$		
	E	I	Z	E	I	Z	E	I	Z
0	$0\ \epsilon^{-j18.1°}$	$.061\ \epsilon^{-j78.9°}$	$0\ \epsilon^{+j60.8°}$	$0\ \epsilon^{-j50.9°}$	$.0633\ \epsilon^{-j111.7°}$	$0\ \epsilon^{+j60.8°}$	$0\ \epsilon^{-j108.1°}$	$.0563\ \epsilon^{-j168.9°}$	$0\ \epsilon^{+j60.8°}$
14.1	$17.55\ \epsilon^{-j17.3°}$	$.0567\ \epsilon^{-j75.9°}$	$312\ \epsilon^{-j56.8°}$	$18.25\ \epsilon^{-j50.1°}$	$.0589\ \epsilon^{-j108.7°}$	$312\ \epsilon^{+j58.6°}$	$16.3\ \epsilon^{-j107.3°}$	$.0524\ \epsilon^{-j165.9°}$	$312\ \epsilon^{+j58.6°}$
28.2	$32.50\ \epsilon^{-j14.2°}$	$.0448\ \epsilon^{-j64.7°}$	$723\ \epsilon^{+j50.5°}$	$33.8\ \epsilon^{-j47°}$	$.0465\ \epsilon^{-j97.5°}$	$723\ \epsilon^{+j50.5°}$	$30.1\ \epsilon^{-j101.2°}$	$.0419\ \epsilon^{-j154.7°}$	$723\ \epsilon^{+j50.5°}$
42.3	$44.20\ \epsilon^{-j8.7°}$	$.034\ \epsilon^{-j37.8°}$	$1295\ \epsilon^{+j28.6°}$	$46.0\ \epsilon^{-j41.5°}$	$.0353\ \epsilon^{-j60.1°}$	$1295\ \epsilon^{+j28.6°}$	$41.0\ \epsilon^{-j98.7°}$	$.0314\ \epsilon^{-j127.3°}$	$1295\ \epsilon^{+j28.6°}$
56.6	$50\ \epsilon^{-j0°}$	$.0328\ \epsilon^{-j11.1°}$	$1525\ \epsilon^{-j11.1°}$	$51.9\ \epsilon^{-j32.8°}$	$.0341\ \epsilon^{-j21.7°}$	$1525\ \epsilon^{-j11.1°}$	$46.3\ \epsilon^{-j90.0°}$	$.0303\ \epsilon^{-j78.9°}$	$1525\ \epsilon^{-j11.1°}$
70.7	.....	.....	.....	$50.9\ \epsilon^{-j19.5°}$	$.0504\ \epsilon^{+j14.4°}$	$1022\ \epsilon^{-j23.8°}$	$45.4\ \epsilon^{-j76.7°}$	$.0447\ \epsilon^{-j42.8°}$	$1022\ \epsilon^{-j23.9°}$
84.8	.....	.....	.....	$50.0\ \epsilon^{-j0°}$	$.0702\ \epsilon^{+j35.5°}$	$723\ \epsilon^{-j35.6°}$	$44.6\ \epsilon^{-j67.2°}$	$.062\ \epsilon^{-j21.6°}$	$723\ \epsilon^{-j35.6°}$
98.9	.....	.....	.....	.....	.....	.....	$44.5\ \epsilon^{-j60°}$	$.077\ \epsilon^{-j6.4°}$	$575\ \epsilon^{-j23.6°}$
113.2	.....	.....	.....	.....	.....	.....	$50.0\ \epsilon^{-j0°}$	$.0888\ \epsilon^{+j11.1°}$	$560\ \epsilon^{-j11.1°}$

S_r	$S = 169.8$ miles $= \frac{3}{4}\lambda$			$S = 226.4$ miles $= \lambda$		
	E	I	Z	E	I	Z
0	$0\ \epsilon^{-j108.1°}$	$.0282\ \epsilon^{-j78.9°}$	$0\ \epsilon^{+j60.8°}$	$0\ \epsilon^{-j288.1°}$	$.0178\ \epsilon^{-j348.9°}$	$0\ \epsilon^{+j60.8°}$
14.1	$8.13\ \epsilon^{-j197.3°}$	$.0261\ \epsilon^{-j255.9°}$	$812\ \epsilon^{+j58.6°}$	$5.182\ \epsilon^{-j287.3°}$	$.0166\ \epsilon^{-j345.9°}$	$312\ \epsilon^{j58.6°}$
28.2	$15.0\ \epsilon^{-j194.3°}$	$.0207\ \epsilon^{-j244.7°}$	$723\ \epsilon^{+j50.5°}$	$9.5\ \epsilon^{-j284.2°}$	$.01315\ \epsilon^{-j334.7°}$	$723\ \epsilon^{j50.5°}$
42.3	$20.4\ \epsilon^{-j188.7}$	$.0157\ \epsilon^{-j217.3°}$	$1295\ \epsilon^{+j28.6°}$	$12.9\ \epsilon^{-j278.7°}$	$.00998\ \epsilon^{-j307.3°}$	$1295\ \epsilon^{j28.6°}$
56.6	$23.1\ \epsilon^{-j180°}$	$.01515\ \epsilon^{-j168.9°}$	$1525\ \epsilon^{-j11.1°}$	$14.68\ \epsilon^{-j270°}$	$.00963\ \epsilon^{-j258.9°}$	$1525\ \epsilon^{-j11.1°}$
70.7	$22.6\ \epsilon^{-j166.7°}$	$.0221\ \epsilon^{-j132.8°}$	$1022\ \epsilon^{-j33.9°}$	$14.52\ \epsilon^{-j256.7°}$	$.0142\ \epsilon^{-j222.9°}$	$1022\ \epsilon^{-j33.9°}$
84.8	$22.2\ \epsilon^{-j147.2°}$	$.0301\ \epsilon^{-j111.6°}$	$723\ \epsilon^{-j35.6°}$	$14.2\ \epsilon^{-j237.2°}$	$.0196\ \epsilon^{-j201.6°}$	$723\ \epsilon^{-j35.6°}$
98.9	$22.19\ \epsilon^{-j120°}$	$.0382\ \epsilon^{-j96.4°}$	$575\ \epsilon^{-j23.6°}$	$14.09\ \epsilon^{-j220°}$	$.0244\ \epsilon^{-j186.4°}$	$575\ \epsilon^{-j33.6°}$
113.2	$25.0\ \epsilon^{-j90°}$	$.0446\ \epsilon^{-j78.9°}$	$560\ \epsilon^{-j11.1°}$	$15.78\ \epsilon^{-j180°}$	$.0282\ \epsilon^{-j168.9°}$	$560\ \epsilon^{-j11.1°}$
127.3	$30.4\ \epsilon^{-j63.3°}$	$.0468\ \epsilon^{-j60.1°}$	$649\ \epsilon^{-j3.2°}$	$19.4\ \epsilon^{-j153.3°}$	$.0300\ \epsilon^{-j150.1°}$	$649\ \epsilon^{-j3.2°}$
141.4	$36.9\ \epsilon^{-j40.7°}$	$.051\ \epsilon^{-j38.3°}$	$723\ \epsilon^{-j2.4°}$	$23.5\ \epsilon^{-j130.7°}$	$.0325\ \epsilon^{-j128.3°}$	$723\ \epsilon^{-j2.4°}$
155.5	$43.6\ \epsilon^{-j20.2°}$	$.055\ \epsilon^{-j13.9°}$	$790\ \epsilon^{-j6.3°}$	$28.3\ \epsilon^{-j110.2°}$	$.0352\ \epsilon^{-j103.9°}$	$795\ \epsilon^{-j6.3°}$
169.8	$50.0\ \epsilon^{j0°}$	$.0627\ \epsilon^{+j11.1°}$	$797\ \epsilon^{-j11.1°}$	$32.1\ \epsilon^{-j90°}$	$.0404\ \epsilon^{-j78.9}$	$795\ \epsilon^{-j11.1°}$
183.9	.....	.....	.....	$35.2\ \epsilon^{-j68.9}$	$.0467\ \epsilon^{-j55.9°}$	$752\ \epsilon^{-j13°}$
198	.....	.....	.....	$39.4\ \epsilon^{-j46.6°}$	$.0543\ \epsilon^{-j32.3°}$	$723\ \epsilon^{-j14.3°}$
212.1	.....	.....	.....	$43.7\ \epsilon^{-j24.3°}$	$.0625\ \epsilon^{-j10.3°}$	$699.2\ \epsilon^{-j14°}$
226.4	.....	.....	.....	$50\ \epsilon^{j0°}$	$.0715\ \epsilon^{+j11.1°}$	$699.3\ \epsilon^{-j11.1°}$

In entries with modulus 0 or ∞, the argument represents a limiting value.

the odd and even quarter wave-lengths, respectively, as measured from the short-circuited end.

The curves marked Z in Figs. 21 to 23, inclusive, are obtained by plotting the values of Z, as calculated in Table 6. They show the variation of the impedance for short-circuited No. 10 B. & S. lines of lengths as indicated, as a function of the distance from either end and also with reference to the voltage and current at

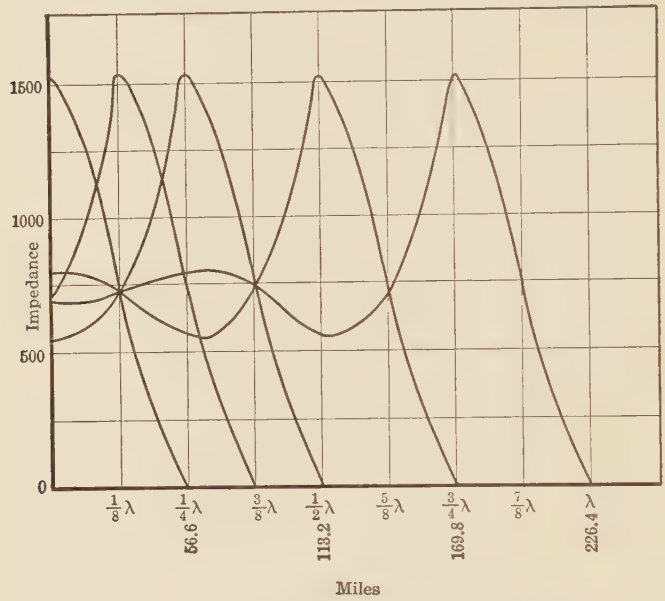

FIG. 26. — Variation of Impedance along No. 10 B. & S. Short-circuited Lines of Different Lengths

corresponding points. The curves in Fig. 26 show the relative variation of impedance with distance as affected by the length of the line.

51. Variation of Phase Angle. — Applying the same method as that used in the determination of the variation of the phase angle along an open-circuited line, we have, by equation (14),

$$\tan \theta = \tan [(\beta_4 - \beta_3) + \alpha]; \tag{15}$$

and since we have

$$\tan (\beta_4 - \beta_3) = \frac{\sin 2\,\iota S_r}{\sinh 2\,\alpha S_r}, \tag{16}$$

we may write

$$\tan\theta = \frac{\sin 2vS_r + \tan\alpha \sinh 2aS_r}{\sinh 2aS_r - \tan\alpha \sin 2vS_r}. \tag{17}$$

For the short-circuited end, where $S_r = 0$, equation (17) becomes

$$\tan\theta = \frac{v + a\tan\alpha}{a - v\tan\alpha}. \tag{18}$$

This last equation shows that, no matter what the length of the line may be, the phase angle at the short-circuited end is always the same, and is independent of the length of the line. In a manner

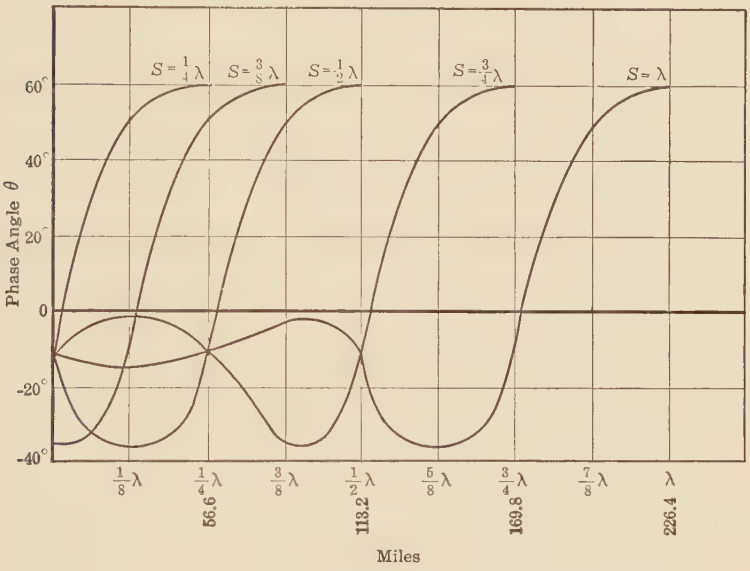

FIG. 27. — Variation of Phase Angle along No. 10 B. & S. Short-circuited Lines of Different Lengths

similar to that shown for the open-circuit line in §42, Chapter VI, it may be proved that $\theta = \alpha_1 = \tan^{-1}(L\omega/R)$; in other words, the phase angle at the end of a short-circuited line depends upon the inductive reactance and resistance per mile of line. For the illustrative No. 10 B. & S. line, the value of θ can never exceed 60.7°, as may be noticed in Fig. 27.

The voltage and the current will obviously be in phase at points along the line where $\theta = 0$, *i.e.*, where

$$\tan(\beta_4 - \beta_3) = -\tan\alpha;$$

or, by equation (16), at points where

$$\tan \alpha = \frac{-\sin 2\,vS_r}{\sinh 2\,aS_r}. \tag{19}$$

By plotting $-\sin (2\,vS_r)/\tan \alpha$ and $\sinh 2\,aS_r$ against S_r to the same scale, the intersection of the curves will give the distance from the short-circuited end at which the voltage and the current are in phase with each other. It will be observed that the first curve is a curve of double frequency, the length of the line determining

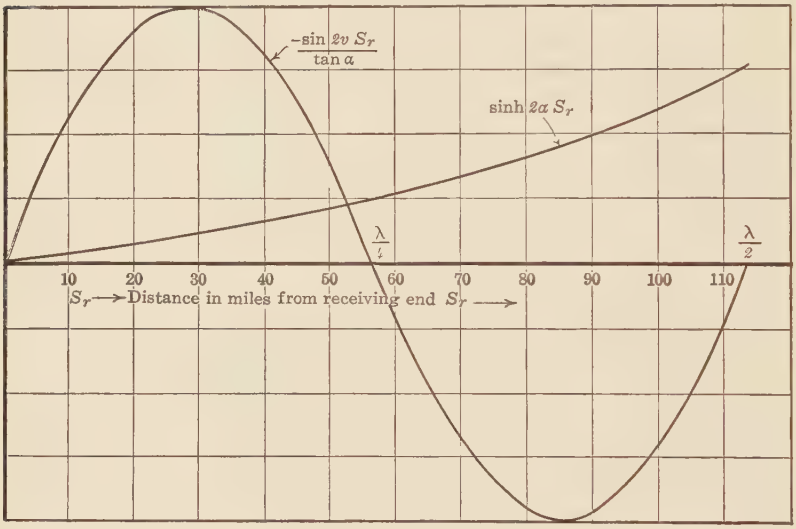
$\frac{-\sin 2v\,S_r}{\tan \alpha}$
$\sinh 2\alpha\,S_r$
$\frac{\lambda}{4}$
$\frac{\lambda}{2}$
10 20 30 40 50 60 70 80 90 100 110
S_r → Distance in miles from receiving end S_r →

Fig. 28

the number of complete alternations. The graphical construction for the No. 10 B. & S. line, obtained from Table 7, and given in Fig. 28, shows that no matter what the line-length may be, there is only one point, about 52 miles from the short-circuited end, at which the current is in phase with the voltage. This may also be seen from the curves showing the variation of phase angle for different lengths of short-circuited lines, given in Fig. 27. For the stretch of line from this point toward the sending end, the current leads the voltage, and for the 52 miles to the short-circuited end, the current lags behind the voltage by a gradually increasing angle, whose maximum is 60.7° at the end of the line.

TABLE 7

S_r	$\frac{-\sin 2vS_r}{\tan \alpha}$	$\sinh 2aS_r$
0	0	0
14.1	3.575	.2608
28.2	5.12	.5381
42.3	3.575	.8523
56.6	0	1.2251
70.7	− 3.575	1.6788
84.8	− 5.12	2.2447
98.9	− 3.575	2.9609
113.2	0	3.8753

CHAPTER VIII

LINE WITH END IMPEDANCE $Z_r e^{j\theta_r}$

The line conditions discussed in the three preceding chapters may be considered as special cases of the more general one in which the line is assumed to be closed at its far end through a receiver of any impedance Z_r with a phase-producing angle of any value θ_r. An open-circuited line may be thought of, for instance, as a line with an infinite impedance at its receiving end, and a short-circuited line, one with an impedance equal to zero at the receiving end. The infinite line, as will appear in the discussion to follow, is equivalent to a line of any length whatever, terminating in an impedance equal to the characteristic line impedance in modulus and in argument.

52. Sending-End Impedance. — Let

Z_r = Impedance of the receiving apparatus at the end of the line,
E_r = Voltage at the end of the line = voltage across impedance Z_r,
I_r = Current in the impedance Z_r;

then

$$Z_r = \frac{E_r}{I_r}. \tag{1}$$

Assuming a line of length S miles, the voltage and the current at the end of the line are, respectively,

$$E_r = E_s \cosh pS - I_s Z_0 \sinh pS, \tag{2}$$

$$I_r = I_s \cosh pS - \frac{E_s}{Z_0} \sinh pS. \tag{3}$$

Dividing equation (2) by equation (3), we get

$$\frac{E_r}{I_r} = Z_r = \frac{E_s \cosh pS - I_s Z_0 \sinh pS}{I_s \cosh pS - \frac{E_s}{Z_0} \sinh pS}, \tag{4}$$

or

$$I_s Z_r \cosh pS - E_s \frac{Z_r}{Z_0} \sinh pS = E_s \cosh pS - I_s Z_0 \sinh pS.$$

Collecting terms, we have

$$I_s(Z_r \cosh pS + Z_0 \sinh pS) = E_s(\cosh pS + \frac{Z_r}{Z_0} \sinh pS).$$

Solving for Z_s, we get

$$(5) \qquad \frac{E_s}{I_s} = Z_s = \frac{Z_r Z_0 \cosh pS + Z_0^2 \sinh pS}{Z_0 \cosh pS + Z_r \sinh pS}.$$

In carrying out the calculation for the sending-end impedance Z_s, the arguments of each of the vector quantities in the preceding expression must be taken in consideration. Inserting these values, we have

$$(6) \quad Z_s\epsilon^{j\theta_s} =$$

$$\frac{Z_r\epsilon^{j\theta_r} Z_0\epsilon^{j\alpha}\sqrt{\cosh^2 aS - \sin^2 vS}\epsilon^{j\beta_1} + Z_0^2\epsilon^{2j\alpha}\sqrt{\cosh^2 aS - \cos^2 vS}\epsilon^{j\beta_2}}{Z_0\epsilon^{j\alpha}\sqrt{\cosh^2 aS - \sin^2 vS}\epsilon^{j\beta_1} + Z_r\epsilon^{j\theta_r}\sqrt{\cosh^2 aS - \cos^2 vS}\epsilon^{j\beta_2}}.$$

If we introduce the abbreviations

$$Z_r Z_0\sqrt{\cosh^2 aS - \sin^2 vS} = A \qquad Z_0\sqrt{\cosh^2 aS - \sin^2 vS} = C$$
$$Z_0^2\sqrt{\cosh^2 aS - \cos^2 vS} = B \qquad Z_r\sqrt{\cosh^2 aS - \cos^2 vS} = D$$

(6) may be written in the form

$$(7) \qquad Z_s\epsilon^{j\theta_s} = \frac{A\epsilon^{j(\theta_r+\beta_1+\alpha)} + B\epsilon^{j(\beta_2+2\alpha)}}{C\epsilon^{j(\beta_1+\alpha)} + D\epsilon^{j(\beta_2+\theta_r)}}.$$

The addition of the vectors in both numerator and denominator may now be effected by throwing the exponentials into trigonometric form.

If we do so, and use the abbreviations

$$(8) \quad \begin{cases} M_1 = A\cos(\theta_r + \beta_1 + \alpha) + B\cos(\beta_2 + 2\alpha), \\ M_2 = C\cos(\beta_1 + \alpha) + D\cos(\beta_2 + \theta_r), \\ N_1 = A\sin(\theta_r + \beta_1 + \alpha) + B\sin(\beta_2 - 2\alpha), \\ N_2 = C\sin(\beta_1 + \alpha) + D\sin(\beta_2 + \theta_r), \end{cases}$$

we may write (7) in the form

$$(9) \qquad Z_s\epsilon^{j\theta_s} = \frac{M_1 + jN_1}{M_2 + jN_2} = \sqrt{\frac{M_1^2 + N_1^2}{M_2^2 + N_2^2}}\epsilon^{j(\theta_1-\theta_2)},$$

where $\theta_1 = \tan^{-1}(N_1/M_1)$, and $\theta_2 = \tan^{-1}(N_2/M_2)$.

To illustrate the method outlined above, let it be required to calculate the sending-end impedance of a No. 10 B. & S. line 98.9 miles long, terminating in a receiving apparatus of impedance equal

to 50 $\epsilon^{j30°}$ ohms. Referring to Table 2 for values of the hyperbolic functions of the complex angles, we have

$$A\epsilon^{j(\beta_1+\theta_r+\alpha)} = 723\ \epsilon^{-j11.1°} \times 50\ \epsilon^{j30°} \times 1.37\ \epsilon^{j\ 162.5°} = 49525\ \epsilon^{j181.4°},$$
$$B\epsilon^{j(\beta_2+2\alpha)} = 723^2\ \epsilon^{-j22.2°} \times 1.089\ \epsilon^{j150°} = 569252\ \epsilon^{j127.8°},$$
$$C\epsilon^{j(\beta_1+\alpha)} = 723\ \epsilon^{-j11.1°} \times 1.37\ \epsilon^{j162.5°} = 991\ \epsilon^{j151.4°},$$
$$D\epsilon^{j(\beta_2+\theta_r)} = 50\ \epsilon^{j30°} \times 1.089\ \epsilon^{j150°} = 54.45\ \epsilon^{j180°},$$

whence, by substituting in equation (7), we get

$$Z_s\epsilon^{j\theta_s} = \frac{49525\ \epsilon^{j181.4°} + 569252\ \epsilon^{j127.8°}}{991\ \epsilon^{j151.4°} + 54.45\ \epsilon^{j180°}},$$

$$\begin{aligned} 49525\ \epsilon^{j181.4°} &= 49525\ (\cos 181.4° + j \sin 181.4°) \\ &= -49510 - j\,1210, \\ 569252\ \epsilon^{j127.8°} &= 569252\ (\cos 127.8° + j \sin 127.8°) \\ &= -348900 + j\,450000, \\ M_1 + jN_1 &= -49510 - 348900 + j\,(450000 - 1210) \\ &= -398410 + j\,448790. \end{aligned}$$

Similarly

$$991\ \epsilon^{j151.4°} = 991\ (\cos 151.4° + j \sin 151.4°) = -870 + j\,473,$$
$$54.45\ \epsilon^{j180°} = 54.45\ (\cos 180° + j \sin 180°) = -54.45,$$
$$M_2 + jN_2 = -870 - 54.45 + j\,473 = -924.45 + j\,473.$$

Hence we have

$$Z_s\epsilon^{j\theta_s} = \frac{-398410 + j\,448790}{-924.45 + j\,473},$$

whence $\theta_1 = 131.6°$ and $\theta_2 = 152.9°$; and

$$Z_s\epsilon^{j\theta_r} = \frac{600000\ \epsilon^{j131.6°}}{1035\ \epsilon^{j152.9°}} = 581\ \epsilon^{-j21.3°} \text{ ohms.}$$

53. Equivalence between an Infinitely Long Line and the same Line of any Length Loaded with an Impedance Equal to the Characteristic Line Impedance. — If the receiving-end impedance is assigned the value infinity, the sending-end impedance, as given by (5), becomes

$$Z_s = \frac{\infty}{\infty},$$

which is indeterminate. If we divide both numerator and denominator of equation (5) by Z_r, however, before assigning to Z_r an infinite value, the expression takes the form

$$Z_s = \frac{Z_0 \cosh pS + \dfrac{Z_0^2}{Z_r} \sinh pS}{\dfrac{Z_0}{Z_r} \cosh pS + \sinh pS}.$$

Now letting Z_r become infinite, we get

$$Z_s = \frac{Z_0}{\tanh pS},$$

which is the sending-end impedance of an open-circuited line as obtained in equation (2), Chapter VI.

On the other hand, if Z_r is equated to zero, equation (5) becomes

$$Z_s = Z_0 \tanh pS,$$

which is the expression for the sending-end impedance of a short-circuited line, as given by equation (3), Chapter VII.

If the receiving apparatus has an impedance equal to the characteristic line impedance, and Z_0 is substituted for Z_r in equation (5), we get

$$Z_s = Z_0.$$

But since the sending-end impedance for an infinite line is equal to the characteristic line impedance, it follows that *if a line of any length whatever terminates in a receiver of impedance equal to the characteristic line impedance, the entire circuit (line and receiver) will behave in a manner similar to an infinite line of that characteristic impedance.* In other words, when $Z_r = Z_0$, the line is equivalent to an infinite line of characteristic impedance Z_0.

54. Sending-End Current. — Since the sending-end current is

$$I_s = \frac{E_s}{Z_s},$$

if we substitute the value of Z_s from (5), we get

$$I_s = E_s \frac{Z_0 \cosh pS + Z_r \sinh pS}{Z_r Z_0 \cosh pS + Z_0^2 \sinh pS}. \tag{10}$$

Obviously, the determination of the numerical value of I_s implies the use of vector quantities and transformations from exponentials to trigonometric forms, as illustrated in the determination of the value of Z_s.

For the 98.9 mile No. 10 B. & S. line, with 50 volts at 796 cycles, impressed at its sending end, and terminating in receiving apparatus of impedance equal to 50 $\epsilon^{j30°}$ ohms, the sending-end current would be

$$I_s \epsilon^{j\phi_s} = \frac{50\ \epsilon^{j0°}}{581\ \epsilon^{-j21.3°}} = .086\ \epsilon^{j21.3°} \text{ amperes.}$$

55. Receiving-End Voltage. — The receiving-end voltage for any transmission line is

$$E_r = E_s \cosh pS - I_s Z_0 \sinh pS.$$

Substituting for I_s its value from (10), we obtain

$$E_r = E_s \cosh pS - \frac{E_s}{Z_0} \frac{Z_0 \cosh pS + Z_r \sinh pS}{Z_r \cosh pS + Z_0 \sinh pS} Z_0 \sinh pS,$$

or, when simplified,

$$E_r = E_s \frac{Z_r}{Z_r \cosh pS + Z_0 \sinh pS}, \tag{11}$$

or, including the arguments of each vector quantity,

$$E_r \epsilon^{j\phi_r} = \frac{E_s Z_r \epsilon^{j\theta_r}}{Z_r \epsilon^{j\theta_r} \sqrt{\cosh^2 aS - \sin^2 vS}\, \epsilon^{j\beta_1} + Z_0 \epsilon^{j\alpha} \sqrt{\cosh^2 aS - \cos^2 vS}\, \epsilon^{j\beta_2}}. \tag{12}$$

For the illustrative No. 10 B. & S. line with an impressed voltage of 50 volts at a frequency of 796 cycles, and a receiver-impedance of 50 $\epsilon^{j30°}$ ohms, we have

$$E_s Z_r \epsilon^{j\theta_r} = 50 \times 50\, \epsilon^{j30°} = 2500\, \epsilon^{j30°},$$
$$Z_r \epsilon^{j\theta_r} \times \cosh pS = 50\, \epsilon^{j30°} \times 1.37\, \epsilon^{j162.5°} = 68.5\, \epsilon^{j192.5°},$$
$$Z_0 \epsilon^{j\alpha} \times \sinh pS = 723\, \epsilon^{-11.1°} \times 1.089\, \epsilon^{j150°} = 787.3\, \epsilon^{j138.9°},$$

whence

$$E_r \epsilon^{j\phi_r} = \frac{2500\, \epsilon^{j30°}}{68.5\, \epsilon^{j192.5°} + 787.3\, \epsilon^{j138.9°}},$$
$$= \frac{2500\, \epsilon^{j30°}}{-659.8 + j\,503.2} = \frac{2500\, \epsilon^{j30°}}{828\, \epsilon^{j142.6°}}$$
$$= 3.019\, \epsilon^{-j112.6°} \text{ volts.}$$

56. Receiving-End Current. — The current in the receiving apparatus at the end of the line is obviously the receiver voltage divided by the receiver impedance:

$$I_r = \frac{E_r}{Z_r}.$$

Substituting the value of the receiver voltage from (11), we have

$$I_r = \frac{E_s}{Z_r \cosh pS + Z_0 \sinh pS}, \tag{13}$$

or, in terms of the receiver voltage,

$$I_r = \frac{E_r \epsilon^{j\phi_r}}{Z_r \epsilon^{j\theta_r}} = \frac{E_r}{Z_r} \epsilon^{j(\phi_r - \theta_r)}. \tag{14}$$

For the No. 10 B. & S. line, with impressed voltage and receiver impedance as in the previous illustrations, the receiver current will be

$$I_r = \frac{3.019\ \epsilon^{-j112.6°}}{50\ \epsilon^{j30°}} = .06039\ \epsilon^{-j142.6°} \text{ amperes.}$$

Obviously, the phase of the current as expressed in the equation, and in the numerical illustration, is with reference to the sending-end voltage, while its phase with respect to the voltage across the receiver is determined only by the relative values of receiver resistance and reactance. In the numerical example it is 30°.

57. Voltage at any Point along a Line with an End Impedance. If the voltage and current at the receiving end are known, the voltage at any point along the line may be calculated from the general equation

$$E = E_r \cosh pS_r + I_r Z_0 \sinh pS_r,$$

where S_r = distance from the receiving end to the point under consideration. Using the exponential form, we get

$$(15)\quad E\epsilon^{j\phi} = E_r\epsilon^{j\phi_r}\sqrt{\cosh^2 aS_r - \sin^2 vS_r}\ \epsilon^{j\beta_3} + I_r\epsilon^{j(\phi_r-\theta_r)}Z_0\epsilon^{j\alpha}\sqrt{\cosh^2 aS_r - \cos^2 vS_r}\ \epsilon^{j\beta_4}.$$

58. Current at any Point along a Line with End Impedance. — In terms of the receiver voltage and distance measured from the receiving end, the current at any point along the line is

$$I = I_r \cosh pS_r + \frac{E_r}{Z_0} \sinh pS_r.$$

Using for E_r and I_r their equivalents as given in equations (12) and (14), and using vector quantities for the hyperbolic functions of the complex angle, we get

$$(16)\quad I = I_r\epsilon^{j(\phi_r-\theta_r)}\sqrt{\cosh^2 aS_r - \sinh^2 vS_r}\epsilon^{j\beta_3} + \frac{E_r\epsilon^{j\phi_r}}{Z_0\epsilon^{j\alpha}}\sqrt{\cosh^2 aS_r - \cos^2 vS_r}\epsilon^{j\beta_4}.$$

59. Impedance at any Point along a Line with End Impedance. — With the voltage and current as calculated by equations (15) and (16), respectively, the impedance at any point along a line that terminates in an end impedance Z_r, may be calculated by dividing the current at the point by the voltage at the point. The argument of that impedance, which represents the phase angle between

the current and the voltage, may be obtained by taking the difference between the arguments of the voltage vector and the current vector for the point under consideration.

60. The Composite Line. — A telephone line is said to be *composite* when the line is made up of sections of different transmission constants. For example, an interurban line connecting two subscribers in two different cities is composite. Such a line usually consists of two rather short terminal underground cable

FIG. 29

sections, joined through a comparatively long aerial line, as shown in Fig. 29.

61. Sending-End Impedance of a Composite Line. — Consider a composite line connecting two subscriber stations A and B, as shown in Fig. 29. Let

$Z_r\epsilon^{j\theta_r}$ = Receiving-end impedance of section 3 at line terminus B,
$Z_{03}\epsilon^{j\alpha_3}$ = Characteristic impedance of section 3,
$Z_{02}\epsilon^{j\alpha_2}$ = Characteristic impedance of section 2,
$Z_{01}\epsilon^{j\alpha_1}$ = Characteristic impedance of section 1.

The impedance at the junction of sections 2 and 3, assuming that A is for the time being the sending-end of the line, may be thought of as the sending-end impedance of section 3 and the receiving-end impedance of section 2; and the impedance at the junction between sections 1 and 2, as the sending end impedance of section 2 and the receiving-end impedance of section 1. Denoting by Z_{23} and Z_{12} the junction impedances with transmission in the direction as indicated by the subscripts, we have

$$Z_{23} = Z_{s3} = Z_{r2}, \quad \text{and} \quad Z_{12} = Z_{s2} = Z_{r1}.$$

The sending-end impedances may now be calculated by (5); this gives

$$(17) \quad Z_{23} = Z_{r2} = Z_{s3} = \frac{Z_r Z_{03} \cosh p_3 S_3 + Z_{03}^2 \sinh p_3 S_3}{Z_{03} \cosh p_3 S_3 + Z_r \sinh p_3 S_3},$$

where p_3 is the propagation constant of section 3, and S_3 is the length of section 3. Similarly,

$$(18) \quad Z_{12} = Z_{r1} = Z_{s2} = \frac{Z_{23}Z_{02}\cosh p_2S_2 + Z_{02}^2 \sinh p_2S_2}{Z_{02}\cosh p_2S_2 + Z_{23}\sinh p_2S_2},$$

where p_2 is the propagation constant of section 2, and S_2 is the length of section 2. Calling Z_{s1} the sending-end impedance of section 1, it will be noted that this is also the sending-end impedance of the entire composite line with transmission in the direction from A to B. The receiving-end impedance of this section is that given by equation (18); hence the value of its sending-end impedance is

$$(19) \quad Z_{sA} = \frac{Z_{12}Z_{01}\cosh p_1S_1 + Z_{01}^2 \sinh p_1S_1}{Z_{01}\cosh p_1S_1 + Z_{12}\sinh p_2S},$$

where p_1 is the propagation constant of section 1, and S_1 is the length of section 1.

It is of great importance to note that *unless sections* 1 *and* 3 *are of the same length and have the same transmission constants, the sending-end impedance at the* B *terminus of the line with transmission from* B *to* A, *under the assumption of equal receiver impedance at the two termini, will differ in magnitude and in argument from the sending-end impedance at* A, *with transmission from* A *to* B.

The sending-end impedance at the B terminus of the composite line, assuming a receiver of impedance $Z_r\epsilon^{j\theta_r}$ at A is

$$(20) \quad Z_{sB} = \frac{Z_{03}Z_{32}\cosh p_3S_3 + Z_{03}^2 \sinh pS}{Z_{03}\cosh p_3S_3 + Z_{32}\sinh p_3S_3},$$

where

$$(21) \quad Z_{32} = \frac{Z_{21}Z_{02}\cosh p_2S_2 + Z_{02}^2 \sinh p_2S_2}{Z_{02}\cosh p_2S_2 + Z_{21}\sinh p_2S_2},$$

and

$$(22) \quad Z_{21} = \frac{Z_rZ_{01}\cosh p_1S_1 + Z_{01}^2 \sinh p_1S_1}{Z_{01}\cosh p_1S_1 + Z_r\sinh p_1S_1}.$$

62. Voltage and Current Values at the Termini and Junctions of a Composite Line. — When the sending-end impedance and junction impedances of a composite line as obtained in the preceding section are known, the potential at the receiving end and those at the various junctions of the line may be calculated by equation (11). Thus the potential at the junction between sections 1 and 2,

which may be considered also as the receiving-end voltage of section 1 and as the sending-end voltage of section 2, is

$$(23) \qquad E_{12} = E_{r1} = E_{s2} = E_{sA} \frac{Z_{12}}{Z_{12} \cosh p_1 S_1 + Z_{01} \sinh p_1 S_1},$$

and the current at that junction

$$(24) \qquad I_{12} = I_{r1} = I_{s2} = \frac{E_{12}}{Z_{12}}.$$

The current at the sending end is

$$(25) \qquad I_{sA} = \frac{E_{sA}}{Z_{sA}}.$$

The potential at the junction between sections 2 and 3, which is also the receiving-end voltage of section 2 and the sending-end voltage of section 3, is

$$(26) \qquad E_{23} = E_{r2} = E_{s3} = E_{12} \frac{Z_{23}}{Z_{23} \cosh p_2 S_2 + Z_{02} \sinh p_2 S_2},$$

and the current at that junction is

$$(27) \qquad I_{23} = I_{r2} = I_{s3} = \frac{E_{23}}{Z_{23}}.$$

The receiving-end voltage at the B terminus of the composite line is

$$(28) \qquad E_B = E_{23} \frac{Z_r}{Z_r \cosh p_3 S_3 + Z_{03} \sinh p_3 S_3},$$

and the receiving-end current is

$$(29) \qquad I_B = \frac{E_B}{Z_r}.$$

To illustrate the above, consider a composite line made up of an underground 5-mile No. 19 B. & S. cable section at the A terminus, and a similar 8-mile cable section at the B terminus of the line, joined by an open No. 10 B. & S. line 70.7 miles long. With a receiver impedance of $300\ \epsilon^{j50°}$ ohms at B, the impedance at the junction of sections 2 and 3, by equation (17), is

$$Z_{23} = \frac{300\ \epsilon^{j50°} \times 571\ \epsilon^{-j42.7°} \cosh p_3 S_3 + 571^2\ \epsilon^{-j85.4°} \sinh p_3 S_3}{571\ \epsilon^{-j42.7°} \cosh p_3 S_3 + 300\ \epsilon^{j50°} \sinh p_3 S_3},$$

where, by Table 1, $571\ \epsilon^{-j42.7°}$ is the characteristic impedance of No. 19 B. & S. line, and

$$\cosh p_3 S_3 = \cosh (.107 + j.111)\ 8 = 1.155\ \epsilon^{j40.7°},$$
$$\sinh p_3 S_3 = \sinh (.107 + j.111)\ 8 = 1.238\ \epsilon^{j61°}.$$

Substituting in the preceding expression and carrying out the operations involved, we get

$$Z_{23} = \frac{198621\ \epsilon^{j48^\circ} + 404133\ \epsilon^{-j24.4^\circ}}{658.5\ \epsilon^{-j2^\circ} + 371\ \epsilon^{j111^\circ}}.$$

Throwing this expression into trigonometric form, adding, and then reconverting into the exponential equivalent, we get

$$Z_{23} = \frac{500000\ \epsilon^{-j2.5^\circ}}{615\ \epsilon^{j31.6^\circ}} = 815\ \epsilon^{-j34.1^\circ} \text{ ohms.}$$

The impedance at the junction between sections 1 and 2 is

$$Z_{12} = \frac{815\ \epsilon^{-j34.1^\circ} \times 723\ \epsilon^{-j11.1^\circ} \times .795\ \epsilon^{j126.1^\circ} + 723^2\ \epsilon^{-j22.2^\circ} \times 1.11\ \epsilon^{j103.3^\circ}}{723\ \epsilon^{-j11.1^\circ} \times .795\ \epsilon^{j126.1^\circ} + 815\ \epsilon^{-j34.1^\circ} \times 1.11\ \epsilon^{j103.3^\circ}},$$

$$= \frac{468000\ \epsilon^{j80.9^\circ} + 580229\ \epsilon^{j81.1^\circ}}{575\ \epsilon^{j115^\circ} + 904.6\ \epsilon^{j69.2^\circ}} = \frac{1048000\ \epsilon^{j81^\circ}}{1360\ \epsilon^{j86^\circ}} = 770\ \epsilon^{-j5^\circ} \text{ ohms.}$$

The sending-end impedance at the A terminus of the composite line is

$$Z_{sA} = \frac{770\ \epsilon^{-j5^\circ} \times 571\ \epsilon^{-j42.7^\circ} \times 1.021\ \epsilon^{j16.9^\circ} + 571^2\ \epsilon^{-j85.4^\circ} \times .772\ \epsilon^{j51.7^\circ}}{571\ \epsilon^{-j42.7^\circ} \times 1.021\ \epsilon^{j16.9^\circ} + 770\ \epsilon^{-j5^\circ} \times 772\ \epsilon^{j51.7^\circ}}$$

$$= \frac{449000\ \epsilon^{-j30.8^\circ} + 250930\ \epsilon^{-j33.7^\circ}}{583\ \epsilon^{-j25.8^\circ} + 595\ \epsilon^{j46.7^\circ}} = \frac{709000\ \epsilon^{-j31^\circ}}{960\ \epsilon^{j10^\circ}} = 738\ \epsilon^{-j41^\circ} \text{ ohms.}$$

With an electromotive force of 50 volts at 796 cycles per sec. impressed at the A terminus of the line, the sending-end current at A is

$$I_{sA} = \frac{E_{sA}}{Z_{sA}} = \frac{50}{738\ \epsilon^{-j41^\circ}} = .0669\ \epsilon^{j41^\circ} \text{ amperes.}$$

By equation (23), the potential at the junction of sections 1 and 2 is

$$E_{12} = E_{r1} = E_{s2} = \frac{50 \times 770\ \epsilon^{-j5^\circ}}{770\ \epsilon^{-j5^\circ} \times 1.021\ \epsilon^{j16.9^\circ} + 571\ \epsilon^{-j42.7^\circ} \times .772\ \epsilon^{j51.7^\circ}}$$

$$= \frac{38500\ \epsilon^{-j5^\circ}}{786\ \epsilon^{j11.9^\circ} + 400\ \epsilon^{j9^\circ}} = \frac{38500\ \epsilon^{-j5^\circ}}{1186\ \epsilon^{j10.9^\circ}} = 32.46\ \epsilon^{-j15.9^\circ} \text{ volts}$$

and the current at that junction is:

$$I_{12} = I_{r1} = I_{s2} = \frac{E_{12}}{Z_{12}} = \frac{32.46\ \epsilon^{-j15.9^\circ}}{770\ \epsilon^{-j5^\circ}} = .042\ \epsilon^{-j10.9^\circ} \text{ amperes}$$

The voltage between junction 2 and 3, by equation (26) is

$$E_{23} = E_{r2} = E_{s3} =$$

$$\frac{32.46\ \epsilon^{-j15.9^\circ} \times 815\ \epsilon^{-j34.1^\circ}}{815\ \epsilon^{-j34.1^\circ} \times .795\ \epsilon^{j126.1^\circ} + 723\ \epsilon^{-j11.1^\circ} \times 1.11\ \epsilon^{j103.3^\circ}}$$

$$= \frac{26464\ \epsilon^{-j50^\circ}}{648\ \epsilon^{j92^\circ} + 802.5\ \epsilon^{j92.2^\circ}} = \frac{26464\ \epsilon^{-j50^\circ}}{1450\ \epsilon^{j92.1^\circ}} = 18.25\ \epsilon^{-j142.1^\circ} \text{ volts.}$$

The current at that junction is

$$I_{23} = I_{r2} = I_{s3} = \frac{E_{23}}{Z_{23}} = \frac{18.25\ \epsilon^{-142.1^\circ}}{815\ \epsilon^{-j34.1^\circ}} = .0223\ \epsilon^{-j108^\circ} \text{ amperes.}$$

The voltage at the B terminus of the composite line, by equation (28), is

$$E_B = E_{r3} = \frac{18.25\ \epsilon^{-j142.1^\circ} \times 300\ \epsilon^{j50^\circ}}{300\ \epsilon^{j50^\circ} \times 1.155\ \epsilon^{j40.7^\circ} + 571\ \epsilon^{-j42.7^\circ} \times 1.238\ \epsilon^{j61^\circ}}$$

$$= \frac{5475\ \epsilon^{-j42.1^\circ}}{346.5\ \epsilon^{j90.7^\circ} + 706.9\ \epsilon^{j18.3^\circ}} = \frac{5475\ \epsilon^{-j42.1^\circ}}{868\ \epsilon^{j40.4^\circ}} = 619\ \epsilon^{-j132.5^\circ} \text{ volts,}$$

and the current at the B terminus is

$$I_\beta = \frac{E_B}{Z_B} = \frac{619\ \epsilon^{-j132.5^\circ}}{300\ \epsilon^{j50^\circ}} = .0206\ \epsilon^{-j182.5^\circ}.$$

CHAPTER IX

ENERGY TRANSFER AND REFLECTION PHENOMENA

63. Physical Conception of Electro-Magnetic Waves. — With a knowledge of the values of and relationships between the voltage and the current along transmission lines with end conditions as discussed in the preceding five chapters, we are in a position to get a physical concept of electro-magnetic waves and the mechanism of their propagation. While we frequently speak of voltage and current waves, the reader must bear in mind that what is really propagated or transferred along a line is energy, partially dissipated as heat through line and load resistance, leakage conductance, and under favorable conditions by direct radiation, and partially distributed and periodically stored in the electric field, resulting in a rise of potential, and in the magnetic field, the result of current flow.

64. Mechanism of Energy Transfer. — To avoid the transients incident on starting at any other value, consider a sinusoidal voltage impressed at the sending end of an infinitely long line at the moment the voltage is at its maximum value. There will be created in the dielectric medium at the point of impression and in its immediate vicinity, an electric field, which is generally thought of as being produced by electric charges, and which manifests itself in a physical state of tension of the medium. This tension is supposed to be the effect of an actual displacement of the electric charges through the medium. With the harmonic variation of the impressed electromotive force, the displacement q will also vary harmonically. The time rate of its change $\pm dq/dt$ causes a harmonically varying current flow, which creates a harmonically varying magnetic field associated with and at right angles to the electric displacement.

The periodic time rate of change of interlinkage of this magnetic field with the conductor, $\pm d\phi/dt$, gives rise in turn to a harmonically varying potential that produces a new electric field at right angles to the magnetic field, *i.e.*, an electric displacement at the neighboring line element, and in the same direction as the original electric displacement. Hence, with a constant supply of the har-

monically varying voltage, the electric and magnetic field will be periodically repeated in space and in time, resulting in what is called an ***electromagnetic wave.***

The transfer of the energy along the line takes place through its successive conversion from the electric state (electrostatic flux) into the magnetic state (magnetic flux) and vice versa, with a gradual displacement in the direction of transfer. Hence the space instantaneous values of voltage and current may be considered as a measure of the electric and magnetic fields, respectively, at any point of the line. The two taken together constitute the electromagnetic wave proper.

65. Standing Waves. — If the line had no resistance, no leakage, and no radiation, the successive periodic conversions of energy from one into the other state would take place with no dissipative loss, and the energy would be divided at any instant of time, and at any point along the line, between the two fields. The attenuation constant a would be zero; the wave-length constant would be

$$v = \omega\sqrt{LC},$$

and the velocity of propagation would be

$$V = \frac{\omega}{v} = \sqrt{\frac{1}{LC}},$$

which was seen to be equal to the velocity of light. The current and voltage would be distributed in time and in space as sine curves. In such a case, the waves representing each are called ***standing waves.*** Such waves may, therefore, be defined as waves in which all the energy is reactive in character, or, in other words, is stored in the line. At any point along the line where the electric field is a maximum, the voltage will be a maximum, and the current and its associated magnetic field will be zero; at any point along the line where the current and the magnetic field are maxima, the potential and the electric field associated with it will be zero. The phase relationship, therefore, between the current and voltage is 90°, as shown in Fig. 30.

Since the conversion of the energy between the two fields is complete, the energy in the electric field is equal to the energy in the magnetic field, *i.e.*,

$$\frac{LI^2}{2} = \frac{CE^2}{2},$$

where E and I are the maximum values of the voltage and the current, and where L and C are the inductance and the capacitance per mile of line, respectively. From the above relation, we see that

$$\frac{E}{I} = \sqrt{\frac{L}{C}}.$$

This same equation, obtained before in quite a different manner, was called ***natural impedance*** or the ***surge impedance.***

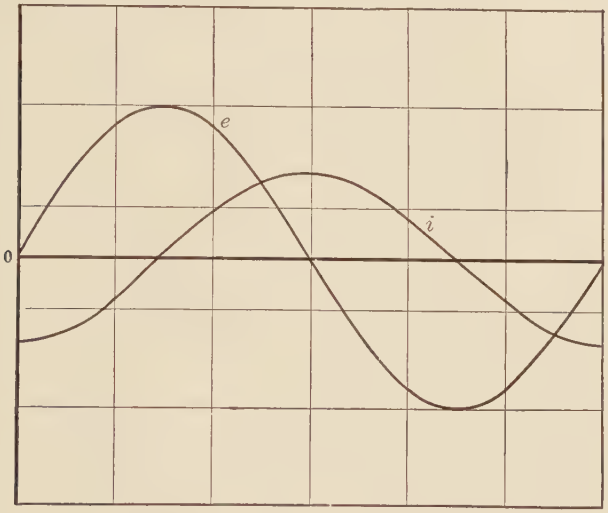

FIG. 30. — Standing Waves

66. Traveling Waves. — Transfer of energy occurs through the agency of the electric and magnetic fields, because distributed line capacitance and inductance are always present. Their absence would make wave transmission impossible. In practical lines having resistance, leakage, and possible radiation, the transfer along the line takes place with a loss, so that less and less energy is converted from one field into the other, as we proceed along the line. This results in what was previously termed *attenuation.*

If the phase angle α_2 produced by the linear line admittance y and the phase angle α_1 produced by the linear line impedance z are equal to one another, *i.e.*, if

$$\frac{L\omega}{R} = \frac{C\omega}{G},$$

and if there are no reflections due to end conditions, as is the case with an infinitely long line, or a line with a receiver impedance equal to the characteristic line impedance, the characteristic impedance is equal to the surge impedance $\sqrt{L/C} = \sqrt{R/G} = Z_0$; the characteristic phase angle α is zero, and the current and voltage are in phase with each other. Under these conditions, while the energy, during its transfer, goes through successive conversions from the electric to the magnetic state and vice versa, all of it is dissipative in character. The wave in such a case is called a ***traveling wave***, which may therefore be defined as a wave transferring only dissipative energy. Such energy transfer obtains when the phase angle at any point along a line is zero, and obviously when the supply is unidirectional.

If we have, on the other hand,

$$\frac{L\omega}{R} \neq \frac{C\omega}{G}$$

and if the line either is infinitely long or terminates in a receiver of impedance equal to the characteristic line impedance in modulus and in argument, the current will lead or lag behind the voltage at all points along the line by the characteristic phase angle. During its transfer, some of the energy is left along the line in gradually decreasing quantities, stored either in the magnetic field if the current lags behind the voltage, or in the electric field if the current leads the voltage. This stored energy is dissipated in the line as a hyperbolically decaying oscillation when the supply at the sending end is removed.

Thus, in our illustrative No. 10 B. & S. infinitely long line, the current leads the voltage by an angle of 11.1°. The dissipative power transferred through the successive conversions from the electric to the magnetic field and vice versa at any point along the line is $EI \cos 11.1°$, while $EI \sin 11.1°$ is the reactive or oscillating component stored as an electric field, E and I being the voltage and current, respectively, at the point under consideration.

Due to energy reflection phenomena resulting from imposed end conditions other than $Z_r = Z_0$, the phase angle between the voltage and the current will change from point to point along the line. At points where the phase angle is zero, all the power transferred is dissipative, while a lagging or leading angle indicates that a portion proportional to the sine of the angle is reactive in character,

a leading angle being manifested by a rise in potential at that point and a lagging angle by a rise in current, in the direction of the transfer of energy.

67. Reflection of Electromagnetic Waves. — In the case of an infinitely long line with uniformly distributed physical constants, both voltage and current are propagated at constant velocity and with uniform rate of attenuation. In the case of a finite line, however, loaded at its end with any definite impedance, the wave, on reaching the end of the line, experiences either a total or a partial arrest, resulting in a nearly complete or partial reversal of the energy transfer, called ***reflection.***

The one exception to this is when the impedance at the end of the line is equal in modulus and in argument to the characteristic line impedance, in which case the line behaves, as proven elsewhere, like an infinite line.

68. Reflections on Open Circuit. — Consider an infinitely long line, whose wave-length constant v, for the frequency of the supplied voltage, is known. The velocity of wave propagation is

$$V = \frac{\omega}{v} \text{ miles/sec.},$$

and the time it will take the wave front to reach a point S miles from the sending end will be

$$t = \frac{S}{V} \text{ seconds.}$$

The values of voltage and current at this distance from the sending end will be

$$E_d = E_s \epsilon^{-pS},$$

and

$$I_d = I_s \epsilon^{-pS-j\alpha},$$

where α is the phase angle between the current and the voltage, and where the subscript d is used to indicate the voltage and the current produced by a forward or direct wave.

The energy per second reaching the point under consideration is $E_d I_d$, of which an amount equal to $E_d I_d \cos \alpha$ is transferred toward the far end of the line as dissipative energy, and $E_d I_d \sin \alpha$ is temporarily stored as an electrostatic field, the angle α being a leading angle.

Imagine now that the line is severed at that point just at the moment when the wave front reaches it, *i.e.*, S/V seconds after the wave has been released at the sending end. The electromagnetic wave has suddenly been arrested in its progress. The current, whose value, if the line were not severed, would have been that given above, and the magnetic field associated with it, are suddenly reduced to zero. All the energy that was temporarily stored in the magnetic field produced by the current at that point, since it is neither transferred not dissipated, is suddenly converted into an electric field. This results in a rise of potential and a further increase of the already leading angle. Since the potential at the open end is now higher than at the point immediately preceding it, a reversed electric displacement will take place; the strain is partially released at a definite rate $-dq/dt$. This results in a reversed current flow, some reduction in potential, and a reversed transfer of energy toward the sending end, until steady conditions are attained.

Since the current at the open end is zero, it is said to be fully reflected with reversal in phase. Using the subscript ρ to indicate reflected values, the current at the end of the now open-circuited line is

$$I_r = I_d + I_\rho = 0,$$

or

$$I_d = -I_\rho.$$

The reflected voltage will depend, however, upon the distance from the sending end at which the line is opened, in other words, upon the length of the open-circuited line. By equation (9), Chapter VI, the voltage at the open-circuited end of a line of length S miles is

$$E_r = \frac{E_s}{\cosh pS}.$$

Comparing this voltage with that which would have existed there, if the line had been continued to infinity, we have

$$\frac{E_r}{E_d} = \frac{E_s}{E_s \epsilon^{-pS} \cosh pS},$$

or, using the exponential equivalent of the hyperbolic function,

$$\frac{E_r}{E_d} = \frac{2}{\epsilon^{-pS}(\epsilon^{pS} + \epsilon^{-pS})} = \frac{2}{1 + \epsilon^{-2pS}}. \tag{1}$$

Hence the voltage at the open end in terms of the voltage that would have been there if the line were infinitely long is

$$E_r = \frac{2}{1 + \epsilon^{-2pS}} E_d. \tag{2}$$

If the open-circuited line is very long, *i.e.*, if the distance S from the sending end at which the infinitely long line is severed is so great that ϵ^{-2pS} is negligibly small, the voltage at the open end is nearly twice the voltage that would have existed there, if the line had been continued to infinity. The voltage wave then is not completely reflected, but may be so assumed if the line is very long.

For values of S for which ϵ^{-2pS} is appreciable, the ratio E_r/E_d is less than 2, and it obviously approaches unity as the length of the line approaches zero.

This discussion shows that even though the wave is fully arrested at the open end of the line, the reversed transfer of the energy is incomplete; a rather large amount remains at the open end as a permanent static strain, as is evidenced by the fact that the phase angle (see (51), Chapter VI) reaches the constant value.

$$\theta = \tan^{-1} \frac{-v + a \tan \alpha}{a + v \tan \alpha}.$$

The voltage may even be higher than the sending-end voltage, if the line is subject to the Ferranti effect.

By equation (2), the voltage at the open-circuited end of a line may be calculated in terms of the voltage at a point of an infinitely long line whose distance from the sending end is equal to the length of the open-circuited line under consideration. Thus the voltage at the open end of a 56.6-mile No. 10 B. & S. line, when the sending-end voltage is 50 volts at a frequency of 796 cycles, may be calculated as follows:

$$E_d = E_s \epsilon^{-pS} = E_s \epsilon^{-aS} \epsilon^{-jvS} = 50 \epsilon^{-.516} \epsilon^{-j90^\circ} = 29.8\, \epsilon^{-j90^\circ},$$

but

$$\epsilon^{-2pS} = \epsilon^{-2 \times .516} \epsilon^{-j2 \times 90^\circ} = .3563\, \epsilon^{-j180^\circ};$$

hence

$$E_r = \frac{2}{1 + \epsilon^{-2pS}} E_d = \frac{2 \times 29.8\, \epsilon^{-j90^\circ}}{1 + .3563\, \epsilon^{-j180^\circ}}$$

$$= \frac{59.6\, \epsilon^{-j90^\circ}}{1 + .3563\, [\cos(-180^\circ) + j \sin(-180^\circ)]}$$

$$= \frac{59.6\, \epsilon^{-j90^\circ}}{1 + .3563\, (-1)} = \frac{59.6\, \epsilon^{-j90^\circ}}{.6437} = 92.7\, \epsilon^{-j90^\circ} \text{ volts.}$$

This result agrees with that obtained in a different manner for the same length of open-circuited line, as shown in Table 4.

In view of what has been said above, the voltage at the open-circuited end of a line may be thought of as the resultant of a direct voltage E_d that would exist there if the line were infinitely long, and a reflected voltage E_d due to the complete arrest of the wave:

$$E_r = E_d + E_\rho, \tag{3}$$

whence the reflected voltage is

$$E_\rho = E_r - E_d.$$

Substituting for E_ρ, its value in terms of E_r, from (2), we find for the reflected voltage the value

$$E_\rho = \left[\frac{2}{1 + \epsilon^{-2pS}} - 1\right] E_d,$$

or

$$E_\rho = \frac{1 - \epsilon^{-2pS}}{1 + \epsilon^{-2pS}} E_d = E_d \tanh pS. \tag{4}$$

The voltage E_r is a measure of the energy stored in the electrostatic field at the open end of the line, and E_ρ is a measure of the energy that is being reflected back into the line toward the sending end, changing the distribution of current, potential, and phase relationship between the two, from that of an infinite line to that of an open-circuited line of finite length, as shown by curves Figs. 9 to 13, Chapter VI.

For the 56.6 mile No. 10 B. & S. line, $E_r = 92.7\ \epsilon^{-j90°}$ volts is a measure of the energy stored in the electric field at the end of the line, and $E_\rho = 62.9\ \epsilon^{-j90°}$ is a measure of the energy transferred back toward the sending end.

69. Reflections on Short Circuit. — Consider a point a distance S_s from the sending end of an infinitely long line. As in the case previously discussed, imagine the line short-circuited at the point just as the wave front reaches it, *i. e.*, S V seconds after the wave has been released at the sending end. The electromagnetic wave has suddenly been arrested in its progress; the voltage, whose value would have been $E_s\epsilon^{-pS}$ if the line had not been short-circuited, is suddenly reduced to zero; the electric field associated with the voltage is also reduced to zero; its rapid rate of decrease $-dq/dt$ results in a large lagging current, and the energy at the point is converted from the electric field into a magnetic field.

The rapid change of interlinkage $-d\phi/dt$ incident to the production of the magnetic field produces in turn a potential higher than at the point immediately preceding the end of the line. This results in a reversal of the electric displacement, and in a back transfer of energy. The electromagnetic wave is arrested with a full reflection of the voltage, since its phase is reversed; and we have

$$E_r = E_d + E_p = 0,$$

or

$$E_d = -E_p.$$

The amount of current reflection will depend, however, upon the distance from the sending end at which the line is short-circuited. By equation (6), Chapter VII, the current at the short-circuited end of a line is

$$I_r = \frac{E_s \epsilon^{-j\alpha}}{Z_0 \sinh pS}.$$

Comparing this current with the current that would have existed there, if the line had been continued to infinity, namely

$$I_d = I_s \epsilon^{-pS-j\alpha} = \frac{E_s}{Z_0} \epsilon^{-pS-j\alpha},$$

we get

$$\frac{I_r}{I_d} = \frac{\epsilon^{j\alpha}}{\epsilon^{-pS-j\alpha} \sinh pS} = \frac{2}{1 - \epsilon^{-2pS}}. \tag{5}$$

As in the case of the open-circuited line with respect to voltage, the current at the end of a very long short-circuited line will be nearly twice the current at that point if the line were continued to infinity, for ϵ^{-2pS} is negligibly small. In other words, the current is never completely reflected. For values of the line-length S such that ϵ^{-2pS} is appreciable, the current at the short-circuited end will be more than $2\,I_d$, and will approach infinity as the line-length approaches zero. The reversed transfer of energy is also incomplete; a rather large proportion remains at the short-circuited end, stored in the magnetic field, as is evidenced by the large current and the constant value of the lagging angle at the end (see (18), Chapter VII):

$$\theta = \tan^{-1} \frac{v + a \tan \alpha}{a - v \tan \alpha}.$$

From what has been said, the current at the short-circuited end may then be considered as the resultant of a current I_d and a

reflected current I_ρ due to the complete arrest of the wave on short-circuit:

$$I_r = I_d + I_\rho ;$$

hence the reflected current is

$$I_\rho = I_r - I_d.$$

Substituting the value of I_r from (5), we get

$$(6) \qquad I_\rho = \left[\frac{2}{1 - \epsilon^{-2pS}} - 1\right] I_d = \frac{1 + \epsilon^{-2pS}}{1 - \epsilon^{-2pS}} I_d = \frac{I_d}{\tanh pS}.$$

A quantity of energy corresponding to the magnetic field produced by I_r remains at the short-circuited end of the line, while an amount of energy corresponding to the magnetic field produced by the reflected current I_ρ is transferred back into the line, changing the distribution of potential, current and phase relation from that on an infinite line, to the distribution pertaining to a line of definite length and short-circuited at its end, as shown by the curves Figs. 21 to 23, Chapter VII.

The short-circuited current I_r and the reflected current I_ρ may be calculated by equations (5) and (6), respectively. Thus, if the 56.6 mile No. 10 B. & S. line is short-circuited at its end, we have

$$I_r = \frac{2}{1 - \epsilon^{-2pS}} I_d,$$

$$I_d = I_s \epsilon^{-pS - j\alpha} = \frac{50\ \epsilon^{-.516 + j90^\circ}}{723\ \epsilon^{-j\,11.1^\circ}} = .0413\ \epsilon^{-j78.9^\circ};$$

hence

$$I_r = \frac{2 \times .0413\ \epsilon^{-j78.9^\circ}}{1 - .3563\ \epsilon^{-j\,180^\circ}} = .061\ \epsilon^{-j78.9^\circ} \text{ amperes.}$$

This result checks with the current at the short-circuited end of a 56.6-mile No. 10 B. & S. line as given in Table 6, calculated in quite a different manner from that given above.

The reflected current, by (6), is

$$I_\rho = \frac{.6437}{1.3563} \times .0413\ \epsilon^{-j78.9^\circ} = .0192\ \epsilon^{-j78.9^\circ} \text{ amperes,}$$

and is a measure of the energy transferred back toward the sending end.

70. Reflections at the End of a Line loaded with an Impedance Z_r; Voltage Transition and Reflection Coefficients. — Imagine that at the instant the wave front of an electromagnetic wave traveling

along an infinite line reaches a point distant S from the sending end, the line is suddenly severed and immediately closed through an impedance $Z_r\epsilon^{j\theta_r}$. From what has been said with regard to the open and short-circuited line, it will appear obvious that the wave will be only partially arrested in its progress; a portion of the energy reaching the point will be reflected back toward the sending end and a portion will be transferred to the end impedance with or without a reactive component, depending upon the phase angle θ_r, *i.e.*, upon the relative values of resistance and reactance of the end impedance. The reactive component will be an electric field if θ_r is a leading angle, and a magnetic field if θ_r is a lagging angle and larger numerically than the characteristic line phase angle α.

If the line were infinitely long, the voltage at the point under consideration would be

$$E_d = E_s\epsilon^{-pS},$$

and, by equation (11), Chapter 8, the receiving potential across the end impedance is

$$E_r = E_s \frac{Z_r}{Z_r \cosh pS + Z_0 \sinh pS},$$

whence the ratio E_r/E_d is

$$\frac{E_r}{E_d} = \frac{Z_r}{\epsilon^{-pS}(Z_r \cosh pS + Z_0 \sinh pS)}. \tag{7}$$

Using the exponential equivalents for the hyperbolic cosine and sine of the complex angle pS, we get

$$\frac{E_r}{E_d} = \frac{2\,Z_r}{\epsilon^{-pS}[Z_r\,(\epsilon^{pS} + \epsilon^{-pS}) + Z_0\,(\epsilon^{pS} - \epsilon^{-pS})]}, \tag{8}$$

or

$$\frac{E_r}{E_d} = \frac{2\,Z_r}{Z_r\,(1 + \epsilon^{-2pS}) + Z_0\,(1 - \epsilon^{-2pS})} = K_{re}. \tag{9}$$

This expression gives the potential across the impedance Z_r, in terms of the voltage E_d that would exist at the point, if instead of the impedance Z_r, the line were continued to infinity.

The ratio K_{re} is called ***voltage transition coefficient.*** Dividing the numerator and the denominator of equation (8) by Z_r, we get:

$$\frac{E_r}{E_d} = K_{re} = \frac{2}{(1 + \epsilon^{-2pS}) + \dfrac{Z_0}{Z_r}\,(1 - \epsilon^{-2pS})},$$

and letting $Z_r = \infty$, we obtain

$$E_r = \frac{2}{1 + \epsilon^{-2pS}}\,E_d.$$

This is the expression for the potential at the open-circuited end of a line, as given by equation (2).

If we let $Z_r = 0$ in expression (9), we get $E_r = 0$, a result previously obtained for the voltage at the short-circuited end of a line. If the impedance at the end of the line is equal to the characteristic line impedance, $Z_r = Z_0$. If we substitute this value in equation (9), we get $E_r = E_d$. As previously stated, this indicates that a line loaded with an impedance equal to the characteristic line impedance behaves like an infinite line, and that no reflection phenomena take place.

At the end of the line assumed loaded with an impedance Z_r, we have $E_r = E_d + E_\rho$. It follows that $E_\rho = E_r - E_d$. Using the value of E_r given by equation (9), we get

$$E_\rho = \left[\frac{2\,Z_r}{Z_r\,(1 + \epsilon^{-2pS}) + Z_0\,(1 - \epsilon^{-2pS})} - 1\right] E_d,$$

or, by multiplying out, we have

$$(10)\quad \frac{E_\rho}{E_d} = \frac{Z_r\,(1 - \epsilon^{-2pS}) - Z_0\,(1 - \epsilon^{-2pS})}{Z_r\,(1 + \epsilon^{-2pS}) + Z_0\,(1 - \epsilon^{-2pS})} = K_{re} - 1 = K_{\rho e}.$$

The ratio $K_{\rho e}$ may be called *voltage reflection coefficient.*

The voltage transition and reflection coefficients, which are vector quantities, enable us to calculate the potential across the end impedance and also the reflected voltage, in terms of the voltage E_d, that would exist at the point, if instead of Z_r, the line were continued to infinity.

Thus for the 56.6 mile No. 10 B. & S. line terminating in impedance equal to $250\ \epsilon^{j30°}$ ohms, the receiver voltage will be, by equation (9)

$$E_r\epsilon^{j\phi} = \frac{2 \times 250\ \epsilon^{j30°} \times 29.8\ \epsilon^{-j90°}}{250\ \epsilon^{j30°}\,(1 + .3563\ \epsilon^{-j180°}) + 723\ \epsilon^{-j11.1°}\,(1 - .3563\ \epsilon^{-j180°})}$$

$$= \frac{500\ \epsilon^{j30°} \times 29.8\ \epsilon^{-j90°}}{250\ \epsilon^{j30°} \times .6437 + 723\ \epsilon^{-j11.1°} \times 1.3563}$$

$$= \frac{500\ \epsilon^{j30°}}{1110\ \epsilon^{-j5.55°}} \times 29.8\ \epsilon^{-j90°} = .45\ \epsilon^{j35.55°} \times 29.8\ \epsilon^{-j90°}$$

$$= 13.41\ \epsilon^{-j54.45°} \text{ volts.}$$

It follows from what precedes that the voltage transition coefficient is $K_{re} = .45\ \epsilon^{j35.55°}$.

The reflected voltage is

$$E_\rho = \left[\frac{500\,\epsilon^{j30°}}{1110\,\epsilon^{-5.55°}} - 1\right] 29.8\,\epsilon^{-j90°} = [.45\,\epsilon^{j35.55°} - 1]\,29.8\,\epsilon^{-j90°}$$
$$= .685\,\epsilon^{j157.6°} \times 29.8\,\epsilon^{-j90°} = 20.4\,\epsilon^{j67.6°}\text{volts.}$$

From what precedes, the voltage reflection coefficient is seen to be $K_{\rho e} = .685\,\epsilon^{j157.6°}$.

71. Current Transition and Reflection Coefficients. — The current transition and reflection coefficients may be obtained in a manner similar to those for the voltage. Thus if, instead of the end impedance Z_r, the line were continued to infinity, the current at that point would be

$$I_d = \frac{E_s}{Z_0}\epsilon^{-pS}.$$

By equation (13), Chapter VIII, the receiving current, when the line is loaded with an end impedance Z_r, is

$$I_r = \frac{E_s}{Z_r \cosh pS + Z_0 \sinh pS}.$$

Hence the ratio of I_r to I_d is

$$(11)\qquad \frac{I_r}{I_d} = \frac{Z_0}{\epsilon^{-pS}(Z_r \cosh pS + Z_0 \sinh pS)}$$
$$= \frac{2\,Z_0}{\epsilon^{-pS}Z_r(\epsilon^{pS} + \epsilon^{-pS}) + Z_0\,(\epsilon^{pS} - \epsilon^{-pS})}$$
$$= \frac{2\,Z_0}{Z_r(1 + \epsilon^{-2pS}) + Z_0(1 - \epsilon^{-2pS})} = K_{rt}.$$

The vector quantity K_{rt} may be called ***current transition coefficient.*** Since $I_r = I_d + I_\rho$, it follows that the reflected current is $I_\rho = I_r - I_d$. Substituting for I_r its value from (11), we get

$$(12)\qquad \frac{I_\rho}{I_d} = \frac{2\,Z_0}{Z_r(1 + \epsilon^{-2pS}) + Z_0(1 - \epsilon^{-2pS})} - 1$$
$$= \frac{Z_0\,(1 + \epsilon^{-2pS}) - Z_r\,(1 + \epsilon^{-2pS})}{Z_r(1 + \epsilon^{-2pS}) + Z_0(1 - \epsilon^{-2pS})} = K_{rt} - 1 = K_{\rho t}.$$

This is a vector quantity, and may be called ***current reflection coefficient.*** By means of the current transition and current reflection coefficients, we can calculate the current in, and the reflected current from any impedance at the end of a line, in terms of the current that would exist at the point if the line were continued to infinity instead of terminating in apparatus.

CHAPTER X

POWER RELATIONS

If E and I are the voltage and the current, respectively, at any point along a line, their product is the power in the line from the point considered onward. This product is called ***voltamperes,*** and will be denoted in the following pages by the symbol $(VA) = EI$.

If the phase angle between the current and the voltage is zero, the power is wholly dissipative in character, either as heat or else as useful power. It is measured in ***watts,*** the voltamperes and watts being numerically equal to one another.

If the phase angle between the current and voltage is 90°, leading or lagging, as it would be in case of a pure dielectric circuit or a pure inductive circuit, respectively, the power is wholly reactive in character, and is stored as a static strain when the angle is a leading angle, or as a magnetic field when the angle is a lagging angle.

If the phase angle is less than 90°, leading or lagging, the power is partly dissipative and partly reactive. The dissipative power is directly proportional to the cosine of the angle, and the reactive power is directly proportional to the sine of the angle of lead or lag.

72. The Infinite Line. — The total power, or energy per second, supplied at the sending end of an infinitely long line, is partially dissipated in the line itself, through its resistance, its leakage conductance, and direct radiation. It is partially stored in the electric field as a static strain, and is further dissipated when the supply at the sending end is removed.

If $(VA)_s$ is the voltamperes supplied at the sending end of the infinite line, the dissipated power in the line is

$$P_R = (VA)_s \cos \alpha, \tag{1}$$

the reactive power is

$$P_X = (VA)_s \sin \alpha, \tag{2}$$

and the total power supplied is

$$(VA)_s = E_s I_s, \tag{3}$$

where the subscripts R and X stand for *dissipative* and *reactive*, respectively.

The voltamperes, the dissipative power, and the reactive power, for any portion of the infinite line, may be calculated from the values of voltage and current at the ends of the portion of the line under consideration. Thus, for a stretch of line S miles from the sending end

(4) Voltamperes supplied $(VA)_s\epsilon^{-j\alpha} = E_sI_s\epsilon^{-j\alpha}$.

The voltamperes at a point S miles from the sending end is

(5) $$(VA)_1\epsilon^{-j\alpha} = E_s\epsilon^{-aS} \times I_s\epsilon^{-aS-j\alpha} = (VA)_s\epsilon^{-2aS-j\alpha},$$

whence the voltamperes for the length of line S miles from the sending end is

(6) $$(VA)_s - (VA)_1 = (VA)_s\epsilon^{-j\alpha} - (VA)_s\epsilon^{-2aS-j\alpha} = (VA)_s\,(1 - \epsilon^{-2aS})\epsilon^{-j\alpha}.$$

The dissipative component of this power is

(7) $$P_R = (VA)_s\,(1 - \epsilon^{-2aS}) \cos\alpha \text{ watts},$$

and the reactive component is

(8) $$P_X = (VA)_s\,(1 - \epsilon^{-2aS}) \sin\alpha \text{ watts},$$

where α is the phase angle between the current and the voltage. For an infinitely long line, α is the same at all points along the line and is equal to the characteristic phase angle of the line.

For our illustrative infinitely long No. 10 B. & S. line with 50 volts at 796 cycles impressed at the sending end, the sending-end voltamperes is

$$(VA)_s\epsilon^{-j\alpha} = 50 \times .0692\,\epsilon^{j11.1^\circ} = 3.46\,\epsilon^{j11.1^\circ} \text{ voltamperes},$$

and the voltamperes at a distance 56.6 miles from the sending end is

$$(VA)_1\epsilon^{-j\alpha} = 50 \times .0692 \times .3563\,\epsilon^{j11.1^\circ} = 1.234\,\epsilon^{j11.1^\circ}.$$

Hence the voltamperes for the 56.5 mile stretch of line is

$$(VA)_s - (VA)_1 = 50 \times .0692\,(1 - .3563)\,\epsilon^{j11.1^\circ} = 2.226\,\epsilon^{j11.1^\circ} \text{ voltamperes},$$

of which the dissipative component is

$$P_R = 2.226 \cos 11.1^\circ = 2.186 \text{ watts},$$

and the reactive component is

$$P_X = 2.226 \sin 11.1^\circ = .4282 \text{ watts}.$$

73. Line with End Impedance $Z_r\epsilon^{j\theta_r}$. — If E_r denotes the potential across the terminals of impedance Z_r, and I_r denotes the current in this impedance, the voltamperes at the terminals of the end impedance $Z_r\,\epsilon^{j\theta_r}$ is

$$(VA)_r = E_r I_r. \tag{8}$$

Substituting in this equation the values of E_r and I_r, from (11) and (13), Chapter VIII, we get

$$(VA)_r = \frac{E_s^2 Z_r}{(Z_r \cosh pS + Z_0 \sinh pS)^2}. \tag{10}$$

Expanding the square in the denominator, dividing both numerator and denominator by Z_r, and assigning respective arguments to the vector quantities involved, (10) may be written in the form

$$(VA)_r = \frac{E_s^2}{Z_r\epsilon^{j\theta_r}\cosh^2 pS\,\epsilon^{j2\beta_1} + \frac{Z_0^2}{Z_r\epsilon^{j\theta_r}}\sinh^2 pS\,\epsilon^{j2\beta_2} + 2\,Z_0\cosh pS\sinh pS\,\epsilon^{j(\beta_1+\beta_2+\alpha)}}.$$

Collecting arguments, this equation becomes

$$(VA)_r = \frac{E_s^2}{Z_r\cosh^2 pS\,\epsilon^{j(2\beta_1+\theta_r)} + \frac{Z_0^2}{Z_r}\sinh^2 pS\,\epsilon^{j(2\beta_2+2\alpha-\theta_r)} + 2\,Z_0\sinh pS\cosh pS\,\epsilon^{j(\beta_1+\beta_2+\alpha)}}.$$

This is the general expression for the voltamperes (11) supplied to the end impedance of a line.

For a line of definite length and fixed electric constants, and with a definite voltage at constant frequency supplied at the sending end, all the quantities in the above equation, with the exception of $Z_r\epsilon^{j\theta_r}$ are constant. Hence the voltamperes will vary only with the numerical value or modulus of the receiver impedance and with its argument θ_r, *i.e.*, with the relative values of its dissipative and reactive components.

74. Variation of Receiver Voltamperes $(VA)_r$ with the Modulus of Receiver Impedance. — The investigation of the variation of the voltamperes with the modulus of the receiver impedance may be carried out most easily by using the reciprocal of the voltamperes. Accordingly equation (11) may be written in the form

$$\frac{1}{(VA)_r} = \frac{Z_r\cosh^2 pS}{E_s^2}\,\epsilon^{j(2\beta_1+\theta_r)} + \frac{Z_0^2\sinh^2 pS}{Z_r E_s^2}\,\epsilon^{j(2\beta_2+2\alpha-\theta_r)} + \frac{2\,Z_0\sinh pS\cosh pS}{E_s^2}\,\epsilon^{j(\beta_1+\beta_2+\alpha)}. \tag{12}$$

Let us assume that the receiver impedance varies so that the ratio of the reactive component to the dissipative component remains constant, *i.e.*, θ_r is a constant. Then all the arguments of the three vectors in expression (12) are constant, and the reciprocal of the voltamperes $1/(VA)_r$ is the sum of three vectors whose position is fixed with respect to one another and with respect to an axis of reference. The first of these vectors varies in length directly with Z_r, the second varies inversely with Z_r, and the third is of fixed length, being independent of Z_r.

The fixed arguments of these three vectors depend upon the transmission constants of the line and its length, and the first two upon the receiver phase angle θ_r. For a line of definite characteristic impedance, the arguments of the three vectors depend only upon the length of line and upon the phase angle θ_r.

75. Variation of Voltamperes with Modulus of Receiver Impedance if $S = n\lambda/4$ and $\theta_r = \alpha$. — We shall discuss in this article the variation of the voltamperes with the modulus of the receiver impedance when the length of the line is an integral number of quarter wave-lengths, and the receiver phase angle is equal to the characteristic line phase angle. Since $S = n\,\lambda/4$ and $\beta_1 = \beta_2$, the arguments of the three vectors will depend only upon the angle θ_r. When $\theta_r = \alpha$, the three vectors in equation (12) are in phase with one another, and the reciprocal of the voltamperes is equal to the algebraic sum of the moduli of the three vectors.

The curve showing the variation of the voltamperes with the numerical value of the receiver impedance when $S = n\,\lambda/4$ and $\theta_r = \alpha$, may then be obtained by assigning increasing values to Z_r in the first two members in the right side of equation (12), adding the numerical values of the three members of that equation, and taking the reciprocal of their sum.

Let us consider, for example, the 56.6-mile No. 10 B. & S. line, with 50 volts at 796 cycles per second impressed at its sending end, and let us assume that the receiver impedance varies from zero to infinity, but that the ratio of its reactive component to its dissipative component remains the same as that for the line, *i.e.*, $\theta_r = \alpha = -\,11.1°$. Then equation (12) becomes

$$\frac{1}{(VA)_r} = \frac{.538^2}{50^2}\,\epsilon^{j180°}\,Z_r\epsilon^{-j11.1°} + \frac{723^2\,\epsilon^{-j22.2°} \times 1.132^2\,\epsilon^{j180°}}{50^2\,Z_r\epsilon^{-j11.1°}} + \frac{2 \times 723\,\epsilon^{-j11.1°} \times 1.132 \times .538\,\epsilon^{j180°}}{50^2}.$$

Multiplying out and collecting terms, we get

$$\frac{1}{(VA)_r} = .000116\, \epsilon^{j168.9°} Z_r + \frac{267.5\, \epsilon^{j168.9°}}{Z_r} + .353\, \epsilon^{j168.9°}.$$

Table 8 gives the values of each term of this equation, the calculated values of $1/(VA)_r$, and the voltamperes for assumed values of Z_r. The letters a, b, and c stand for the numerical constants of the three terms, respectively.

TABLE 8

Z_r	aZ_r	$\frac{b}{Z_r}$	c	$\frac{1}{(VA)_r}$	$(VA)_r$
0	0	∞	.353	∞	0
250	.029	1.07	"	1.452	.690
500	.058	.536	"	.947	1.056
750	.087	.357	"	.797	1.252
1000	.116	.268	"	.737	1.356
1250	.145	.214	"	.712	1.404
1500	.174	.170	"	.706	1.416
1750	.203	.153	"	.709	1.410
2000	.232	1.34	"	.719	1.391
2500	.29	.117	"	.750	1.332
3000	.348	.089	"	.790	1.266
3500	.406	.079	"	.835	1.190
4000	.464	.067	"	.884	1.136
....	....	....	"	....	....
....	....	....	"	....	....
∞	∞	0		∞	0

Curve A in Fig. 31, which shows the variation of the voltamperes with the modulus of the receiver impedance, is plotted from the data given in the Table 8. It will be noted that the voltamperes takes on the same value twice, *i.e.*, for two values of Z_r. Its value is 0 for $Z_r = 0$ and also for $Z_r = \infty$. This might have been expected, since in the first case the line is short-circuited and in the second case the line is open-circuited.

The voltamperes reach a maximum for a definite value of receiver impedance, and there are two values of Z_r for which the voltamperes are the same as at a point along an infinite line whose distance from the sending end is equal to the length of the line under consideration, one of which is, obviously, $Z_r = Z_0$.

Curve B of Fig. 31 will be referred to later.

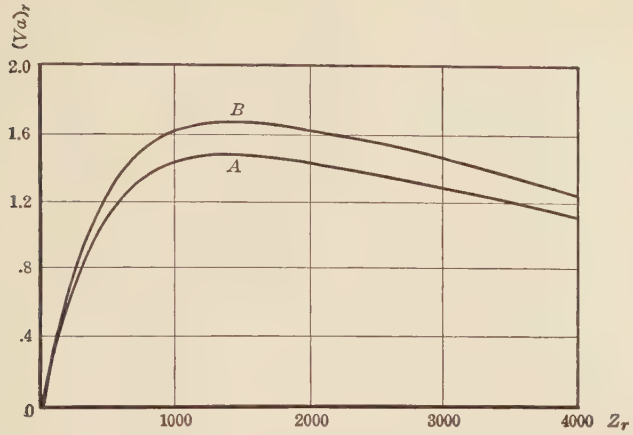

FIG. 31. — Variation of $(Va)_r$ with Z_r
θ constant; $S = 56.6$ miles; $E_s = 50$ volts

76. Variation of Voltamperes with the Modulus of the Receiver Impedance when $S \neq n\lambda/4$, or when $\theta \neq \alpha$. — For lengths of line other than an integral number of quarter wave-lengths or when the argument of the receiver impedance is not equal to the

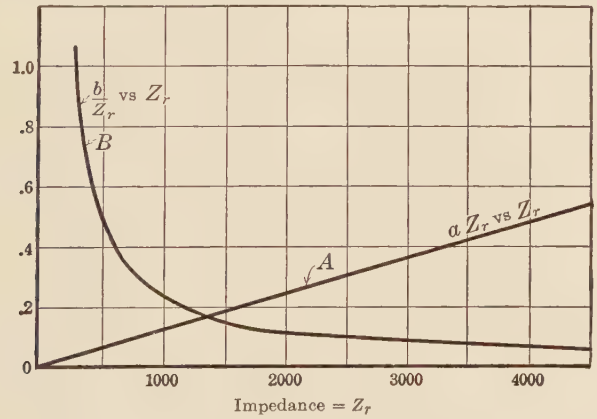

FIG. 32. — Variation of aZ_r and $\frac{b}{Z_r}$ with Z_r

characteristic phase angle, the three vectors in equation (12) are not in phase with each other, and the value of $1/(VA)_r$ must be obtained by vector addition of these quantities. The law of varia-

tion of the voltamperes with the modulus of the receiver impedance when the argument θ_r is constant, may be obtained, however, by a graphical construction based on the vector quantities of equation (12).

Figure 32 gives the construction for the 56.6 miles No. 10 B. & S. line with 50 volts at a frequency of 796 cycles per second, terminating in an impedance whose numerical value varies from zero to

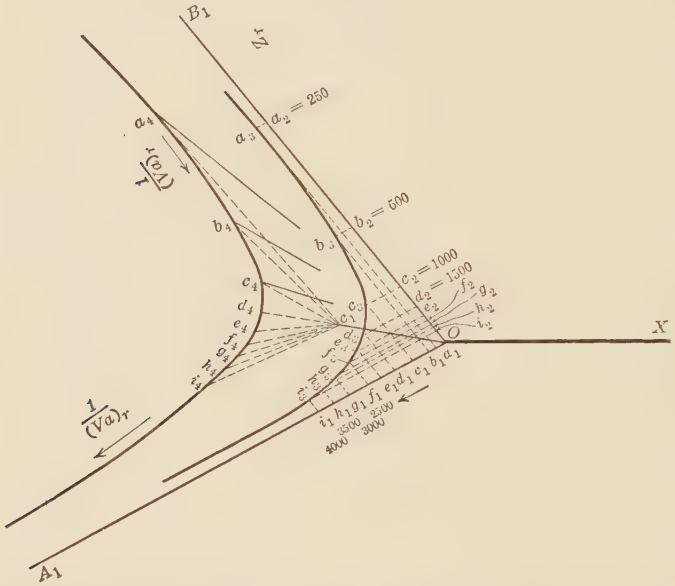

Fig. 33. — Graphical Determination of the Variation of $1/(Va)$ with Z_r. $\theta = 30$; $S = 56.6$ miles; No. 10 B. & S. line

infinity, but whose argument is maintained constant at $\theta_r = 30°$. The graphical construction is as follows.

(1) Plot against Z_r the modulus of the first vector of equation (12),

$$\frac{\cosh^2 pS}{E_s^2} Z_r.$$

This will give a straight line, as shown by curve marked A, Fig. 32.

(2) To the same scale as curve A, plot against Z_r the modulus of the second vector term of equation (12),

$$\frac{Z^2{}_0 \sinh^2 pS}{E_s^2 Z_r}.$$

This will give a rectangular hyperbola, as shown by curve marked B, Fig. 32.

(3) Since the line is assumed fixed in length, and θ_r is constant for all values of Z_r, all the ordinates of curve A have the argument $2\beta_1 + \theta_r$. Hence they may be laid off along a line OA_1, Fig. 33, displaced from any reference axis OX by an angle $2\beta_1 + \theta_r$.

Similarly, all the ordinates of curve B, Fig. 32, have the same argument $2\beta_2 + 2\alpha - \theta_r$, and may also be laid off along a line OB_1, Fig. 33, displaced from the reference axis OX by an angle $2\beta_2 + 2\alpha - \theta_r$.

Values Oa_1, Ob_1, Oc_1, etc., along the line OA_1 and values Oa_2, Ob_2, Oc_2, etc., along the line OB_1, corresponding to the same values of Z_r and obtained from the curves A and B, respectively, Fig. 32, are added graphically, resulting in the vectors Oa_3, Ob_3, Oc_3, etc. A curve drawn through points Oa_3, Ob_3, Oc_3, etc., will give the values of

$$\frac{Z_r \cosh^2 pS}{E_s^2}\,\epsilon^{j(2\beta_1+\theta_r)} + \frac{Z_0^2 \sinh^2 pS}{Z_r E_s^2}\,\epsilon^{j(2\beta_2+2\alpha-\theta_r)},$$

as Z_r is increased indefinitely, the curve being, obviously, asymptotic to both OA_1 and OB_1, as may be observed from Fig. 33.

(4) To the same scale as the vectors OA_1 and OB_1, lay off vector OC_1 equal to the third term of (12),

$$\frac{2 Z_0 \sinh pS \cosh pS}{E_s^2},$$

making an angle with the axis of reference OX, equal to $\beta_1 + \beta_2 - \alpha$, and add it graphically to Oa_3, Ob_3, Oc_3, etc. The resulting vectors, Oa_4, Ob_4, Oc_4, etc., are the values of $1/(VA)_r$ as Z_r is increased indefinitely. The voltamperes may now be obtained by taking the reciprocal of $1/(VA)_r$ corresponding to the particular value of Z_r under consideration.

For the given construction, the numerical values of the moduli of the three vectors for increasing values of Z_r, are taken from Table 8. The phase angle of the variable vector OA_1 is

$$2\beta_1 + \theta_r = 180° + 30° = 210°$$

and the phase angles of the vectors OB_1 and OC_1 are, respectively,

$$2\beta_2 + 2\alpha - \theta_r = 180° - 22.2° - 30° = 127.8°,$$
$$\beta_1 + \beta_2 + \alpha = 90° + 90° - 11.1° = 168.9°.$$

Curve B, Fig. 31, shows the variation of the voltamperes with Z_r as obtained in the manner described above. It will be observed that this curve is of the same shape as the one obtained for the case when $\theta_r = \alpha$, there being a definite value of Z_r for which the voltamperes is the same as for a point along an infinite line, whose distance from the sending end is equal to the length of the line under consideration.

77. Receiver Impedance for Maximum Voltamperes. Terminal Gain and Loss. — The value of Z_r for which the voltamperes will

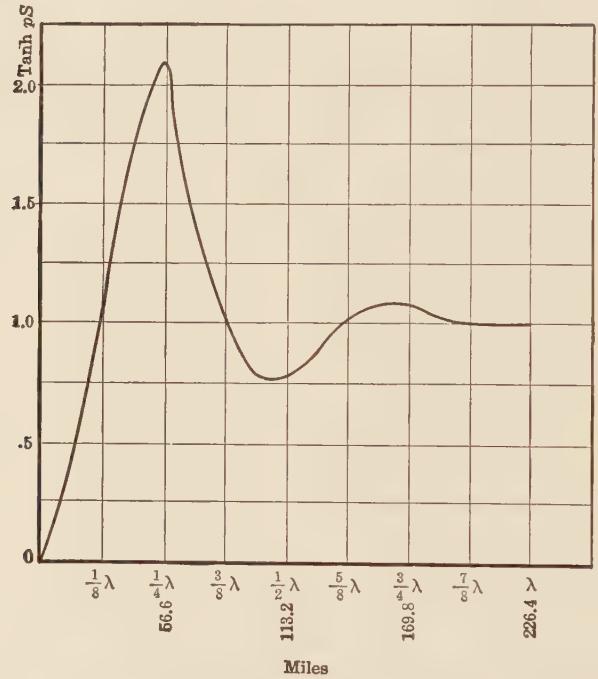

FIG. 34. — Variation of tanh pS with Length of No. 10 B. & S. Line

be a maximum is obviously that which will make $1/(VA)_r$ a minimum. This may be obtained by differentiating $(VA)_r$ from equation (10) with respect to Z_r, equating the derivative to zero, and solving for Z_r. This gives

$$Z_{rm} = Z_0 \frac{\sinh pS}{\cosh pS} = Z_0 \tanh pS. \tag{13}$$

This equation indicates that for a line of definite length, the maximum voltamperes will be obtained when the ratio of the receiver impedance to the characteristic line impedance is equal to the hyperbolic tangent of the complex angle pS. Figure 34 shows the variation of tanh ps with line length, for our illustrative No. 10 B. & S. line. An inspection of this curve with reference to the above statement indicates that the impedance Z_{rm} for maximum voltamperes may be smaller or larger than the characteristic line impedance, depending upon the length of line. For lines equal to any odd number of eighths of a wave-length, however, since tanh $pS = 1$, the maximum voltamperes is obtained when the modulus of the receiver impedance is equal to the modulus of the characteristic impedance, $Z_{rm} = Z_0$. That this must be so, may be seen also from the fact that

$$\tanh pS = \sqrt{\frac{\cosh^2 aS - \cos^2 vS}{\cosh^2 aS - \sin^2 vS}} = 1$$

when $\cos vS = \sin vS$, *i.e.*, when

$$vS = (2n - 1)\,\pi/4 = (2n - 1)\,\lambda/8,$$

where n is any integer.

Moreover, since for long lines both $\cos vS$ and $\sin vS$ are insignificant in comparison with $\cosh aS$, the receiver impedance for maximum voltamperes, for very long lines, is very nearly equal to the characteristic line impedance.

For any line for which $Z_{rm} \neq Z_0$, there always will be two values of receiver impedance for which the voltamperes will be the same as at a point along an infinite line of like characteristic impedance, whose distance from the sending-end is equal to the length of the line under consideration, *i.e.*,

$$(VA)_r = \frac{E_s^2\,\epsilon^{-2pS}}{Z_0\,\epsilon^{j\alpha}} = \frac{E_s^2\,\epsilon^{-2aS}}{Z_0}\,\epsilon^{-j(2vS+\alpha)}.$$

The two values of Z_r may be obtained by substituting this value of $(VA)_r$ in equation (12), and solving for Z_r. This gives

$$(14) \qquad \frac{Z_0\,\epsilon^{-j\theta_1}}{E_s^2\,\epsilon^{-2aS}} = \frac{Z_r \cosh^2 pS}{E_s^2}\,\epsilon^{j\theta_2} + \frac{Z_0^2 \sinh^2 pS}{Z_r E_s^2}\,\epsilon^{j\theta_3} + \frac{2\,Z_0 \sinh pS \cosh pS}{E_s^2}\,\epsilon^{j\theta_4}$$

where $\theta_1 = 2\,vS + \alpha$, $\theta_2 = 2\,\beta_1 + \theta_r$, $\theta_3 = 2\,\beta_2 + 2\,\alpha - \theta_r$, and $\theta_4 = \beta_1 + \beta_2 + \alpha$.

The constant E_s^2, which appears in the denominator on both sides of the equality sign, may be canceled. If we separate each vector into its real and imaginary components, equation (14) is also separated into the two equations

$$(15)\quad \frac{Z_0}{\epsilon^{-2\alpha S}}\cos\theta_1 = Z_r\cosh^2 pS\cos\theta_2 + \frac{Z_0^2}{Z_r}\sinh^2 pS\cos\theta_3 + 2\,Z_0\sinh pS\cosh pS\cos\theta_4,$$

and

$$(16)\quad \frac{Z_0}{\epsilon^{-2\alpha S}}\sin\theta_1 = Z_r\cosh^2 pS\sin\theta_2 + \frac{Z_0^2}{Z_r}\sinh^2 pS\sin\theta_3 + 2\,Z_0\sinh pS\cosh pS\sin\theta_4.$$

Either of these equations may be used for the determination of the receiver impedance that would result in the same value of voltamperes as at a point along an infinite line distant from the sending end equal to the length of line under consideration, *i.e.*, for the condition

$$(VA)_r = \frac{E_s\,\epsilon^{-2pS}}{Z_0\,\epsilon^{j\alpha}},$$

Using equation (15), and letting for brevity's sake

$$\cosh^2 pS\cos\theta_2 = A,$$
$$Z_0\sinh^2 pS\cos\theta_3 = B,$$
$$2\,Z_0\sinh pS\cosh pS\cos\theta_4 = C,$$
$$\frac{Z_0}{\epsilon^{-2\alpha S}}\cos\theta_1 = D,$$

we may write equation (15) in the form

$$D = AZ_r + \frac{B}{Z_r} + C,$$

or

$$AZ_r^2 + (C - D)\,Z_r + B = 0.$$

This is a quadratic equation, and its solution is

$$(17)\qquad Z_r = \frac{-(C-D) \pm \sqrt{(C-D)^2 - 4\,AB}}{2\,A}.$$

For the values of Z_r given by this expression, the voltamperes at the receiving end will be the same as at a point along an infinite line of like characteristic impedance, whose distance from the

sending end is equal to the length of the line under consideration, the two lines having the same transmission constants.

As an illustration of the discussion given above, consider a 42.3 mile No. 10 B. & S. line. The numerical value of the receiver impedance that would produce the maximum voltamperes, by equation (13), is

$$Z_{rm} = Z_0 \tanh pS = 723 \times 1.795 = 1297.8 \text{ ohms.}$$

The impedance of fixed argument $\theta_r = 30°$, for which the voltamperes will be equal to the voltamperes obtainable at a point 42.3 miles from the sending end of an infinitely long No. 10 B. & S. line, may be obtained from equation (17). For our example, the values to be substituted in (17) are, by Table 2,

$$\theta_1 = 2 \times 67.5° - 11.1° = 123.9°,$$
$$\theta_2 = 2 \times 41.6° + 30° = 113.2°,$$
$$\theta_3 = 2 \times 81.3° - (2 \times 11.1°) - 30° = 110.4°,$$
$$\theta_4 = 41.6° + 81.3° - 11.1° = 111.8°,$$

$$\begin{aligned} A &= \cosh^2 pS \cos \theta_2 \\ &= .557^2 \cos 113.2° \\ &= -.1224, \\ B &= Z_0^2 \sinh^2 pS \cos \theta_3 \\ &= 723^2 \times 1.00 \times \cos 110.4° \\ &= -182500, \\ C &= 2 Z_0 \sinh pS \cosh pS \cos \theta_4 \\ &= 2 \times 723 \times 1\ 00 \times .557 \cos 111.8° \\ &= -305, \\ D &= \frac{Z_0}{\epsilon^{-2aS}} \cos \theta_1 \\ &= \frac{723}{\epsilon^{-.774}} \cos 123.9° \\ &= -873. \end{aligned}$$

Hence the value of Z_r, by (17), is

$$Z_r = \frac{-(-305 + 873) \pm \sqrt{(-305 + 873)^2 - 4 \times .1224 \times 182500,}}{-2 \times .1224}$$

and the two values of Z_r are

$$Z_{r1} = \frac{-568 + 492.2}{-.2448} = 309\, \epsilon^{j30°} \text{ ohms,}$$

and

$$Z_{r2} = \frac{-568 - 492.2}{-.2448} = 4330\, \epsilon^{j30°} \text{ ohms.}$$

Either of these two receiving impedances will produce for $(VA)_r$ the value

$$(VA)_r = \frac{E_s^2 \epsilon^{-2\alpha S}}{Z_0}.$$

If the sending-end voltage is 50 volts, the receiving-end voltamperes will be

$$(VA)_r = \frac{2500 \times .4613}{723} = 1.593 \text{ voltamperes.}$$

For all values of impedance between 309 and 4330 ohms, the voltamperes will be larger than that obtainable with an impedance equal to $Z_r = Z_0 = 723$, *i.e.*, larger than 1.593 voltamperes, the maximum occurring when $Z_r = 1297.8$ ohms.

All values of impedance between these two limiting values are said to produce a *terminal gain.* Receiver impedances less than 309 ohms or larger than 4330 ohms will produce receiving-end voltamperes less than that obtainable with a receiver impedance equal to the chracteristic line impedance and are said to produce a *terminal loss.*

An interesting case with reference to the above is when the length of the line is a quarter wave-length or any integral multiple of that length, and when the argument of the variable receiver impedance is constant at a value equal to the characteristic line angle α. As stated in § 75, in which power relations for such a line are discussed, the vectors in equation (12) are in phase with one another, and $1/(VA)_r$ is the algebraic sum of the moduli of the three vectors. To find the receiver impedance for which the voltamperes will be the same as at a point along an infinite line whose distance from the sending end is equal to the length of the line under consideration (one quarter wave-length or an integral number of quarter wave-lengths), equation (14) may be written in the form

$$(18) \quad \frac{Z_0}{\epsilon^{-2pS}} = Z_r \cosh^2 pS + \frac{Z_0^2 \sinh^2 pS}{Z_r} + 2 Z_0 \sinh pS \cosh pS.$$

If we set, for brevity,

$$\cosh^2 pS = A,$$
$$Z_0^2 \sinh^2 pS = B,$$
$$2 Z_0 \sinh pS \cosh pS = C,$$
$$\frac{Z_0}{\epsilon^{-2pS}} = D,$$

we may write equation (18) in the form

$$AZ_r^2 + (C - D)\,Z_r + B = 0.$$

This is a quadratic equation whose solution is

$$(19) \qquad Z_r = \frac{-(C - D) \pm \sqrt{(C - D)^2 - 4\,AB}}{2\,A}.$$

By the definitions of C and D, we have

$$\begin{aligned} C - D &= 2\,Z_0 \sinh pS \cosh pS - \frac{Z_0}{\epsilon^{-2pS}} \\ &= Z_0 \left(\sinh 2\,pS - \frac{1}{\cosh 2\,pS - \sinh 2\,pS}\right) \\ &= Z_0 \frac{\sinh 2\,pS \cosh 2\,pS - \sinh^2 2\,pS - 1}{\cosh 2\,pS - \sinh 2\,pS}, \end{aligned}$$

or, since $\sinh^2 2\,pS = \cosh^2 2\,pS - 1$,

$$\begin{aligned} C - D &= Z_0 \frac{\sinh 2\,pS \cosh 2\,pS - \cosh^2 2\,pS}{\cosh 2\,pS - \sinh 2\,pS} \\ &= Z_0 \frac{-\cosh 2\,pS\,(\cosh 2\,pS - \sinh 2\,pS)}{\cosh 2\,pS - \sinh 2\,pS} \\ &= -Z_0 \cosh 2\,pS \end{aligned}$$

By the definitions of A and B, we have

$$\begin{aligned} AB &= \cosh^2 pS\, Z_0^2 \sinh^2 pS \\ &= (Z_0 \sinh pS \cosh pS)^2 \\ &= \frac{Z_0^2 \sinh^2 2\,pS}{4}, \end{aligned}$$

whence

$$4\,AB = Z_0^2 \sinh^2 2\,pS,$$

and

$$(C - D)^2 - 4\,AB = Z_0^2 \cosh^2 2\,pS - Z_0^2 \sinh^2 2\,pS = Z_0^2,$$

so that

$$\sqrt{(C - D)^2 - 4\,AB} = Z_0.$$

Substituting these values in equation (19), we get

$$(20) \qquad Z_r = \frac{Z_0 \cosh 2\,pS \pm Z_0}{2 \cosh^2 pS} = Z_0 \frac{\cosh 2\,pS \pm 1}{2 \cosh^2 pS},$$

or, since

$$\cosh 2\,pS = \cosh^2 pS + \sinh^2 pS \text{ and } \cosh^2 pS - \sinh^2 pS = 1,$$

$$Z_r = Z_0 \frac{(\cosh^2 pS + \sinh^2 pS) \pm (\cosh^2 pS - \sinh^2 pS)}{2 \cosh^2 pS}.$$

Hence the two solutions of equation (19) are

$$Z_{r1} = Z_0, \tag{21}$$

and

$$Z_{r2} = Z_0 \frac{\sinh^2 pS}{\cosh^2 pS} = Z_0 \tanh^2 pS. \tag{22}$$

It follows from what precedes that for a line whose length is a quarter wave-length or an integral multiple of that length, the receiver impedance for which the voltamperes will be the same as at a point along an infinite line whose distance from the sending end is a quarter wave-length, or any number of quarter wave-lengths, are Z_0 and $Z_0 \tanh^2 pS$, provided that the receiver phase angle is equal to the characteristic line phase angle.

For a line whose length is an odd number of eighths of a wave-length, and with $\theta_r = \alpha$, there is only one solution, $Z_r = Z_0$, for which the voltamperes are equal to the voltamperes obtainable at a point along an infinite line whose distance from the sending end is an odd number of eighths of a wave-length. This particular value of the receiver impedance, equal to the characteristic line impedance, will produce the maximum voltamperes (see (13)), since for such a line, $\tanh pS = 1$. This is also approximately true for very long lines.

Considering the variation of the terminal voltamperes with the modulus of the receiver impedance for any definite value of receiver phase angle θ_r, we may conclude from the preceding discussion that

(1) The maximum value of terminal voltamperes is obtained when $Z_r = Z_0 \tanh pS$.

(2) This value of Z_r may be larger or smaller than Z_0, depending upon the length of the line.

(3) For a line whose length is an eighth of a wave-length, or an odd multiple of that length, and approximately for very long lines, the receiver impedance that will produce the maximum voltamperes must be equal in modulus to the characteristic line impedance.

(4) For lines of such length that the receiver impedance for maximum voltamperes, Z_{rm}, is larger than or smaller than the characteristic line impedance, and with θ_r of any value whatever, the voltamperes for all values of Z_r within the limiting values of the solutions of equation (17), will be larger than that obtainable

with a receiver impedance equal to $Z_0\epsilon^{j\alpha}$. Such lines are said to produce a *terminal gain*, while the voltamperes for all other values of Z_r will be smaller than that obtainable with a receiver impedance equal to $Z_0\epsilon^{j\alpha}$, and are said to produce a *terminal loss*.

(5) For a line whose length is a quarter wave-length, or any number of quarter wave-lengths, and with receiver phase angle equal to the characteristic line phase angle, there will be a *terminal gain* for all values of receiver impedance between

$$Z_r = Z_0 \quad \text{and} \quad Z_r = Z_0 \tanh^2 pS,$$

and a *terminal loss* if

$$Z_r < Z_0, \quad \text{or if} \quad Z_r > Z_0 \tanh^2 ps.$$

(6) For a line whose length is an eighth of a wave-length, or any odd multiple of that length, and also for very long lines, no terminal gain can be obtained by a variation of the modulus of the receiver impedance.

78. Variation of Receiver Voltamperes with Receiver Phase Angle. — The variation of the receiver voltamperes with changes in the relative values of the dissipative and reactive components of the receiver impedance, when the modulus of their vector sum, *i.e.*, the numerical value of the receiver impedance, remains constant, may be obtained by a graphical construction similar in principle to the one used for the determination of the law of variation of the voltamperes with the modulus of the receiver impedance. In the latter case, the vector members of equation (12) depending upon Z_r vary in length and are fixed with respect to each other and with respect to the axis of reference, while in the case here considered, they remain constant in length, but change their position with respect to each other and also with respect to the axis of reference.

From equation (12), we see that the vector of constant length, $Z_r \cosh^2 pS$, will change its position from $2\beta_1 - 90°$ (Oa_1, Fig. 35) to $2\beta_1 + 90°$ (Og_1, Fig. 35) at the same rate as the vector of constant length $(Z^2{}_0/Z_r) \sinh^2 pS$ changes its position from $2\beta_2 + 2\alpha + 90°$ (Oa_2, Fig. 35) to $2\beta_2 + 2\alpha - 90°$ (Og_2, Fig. 35).

This change in vector position can be brought about by varying the reactive and dissipative components of Z_r in such a manner that the argument θ_r varies from $-90°$ to $90°$, *i.e.*, Z_r changes from a pure dielectric reactance $1/(C\omega) = Z_r$, through a pure non-inductive resistance $R = Z_r$ to a pure inductive reactance $L\omega = Z_r$.

Referring to the construction given in Fig. 35, which is for a 56.6-mile No. 10 B. & S. line, with a sending-end voltage of 50 volts at 796 cycles per second, and feeding into a receiver impedance of 1000 ohms, whose phase angle varies from $-90°$ to $+90°$, we see that the variation of $1/(VA)_r$ with θ_r, is obtained as follows.

(1) Starting with $Z_r \cosh^2 pS = Oa_1$, making an angle $2\beta_1 - 90°$ with the axis of reference Ox, when $\theta_r = -90°$, and with $(Z_0^2/Z_r) \sinh^2 pS = Oa_2$ making an angle $2\beta_2 + 2\alpha + 90°$ with the axis of reference Ox, we find that their vector sum $Oa_1 + Oa_2$ will be Oa_3. As θ_r changes its value from $-90°$ to $-60°$, the line Oa_1 moves away from its original position through an angle of 30° to the new position Ob_1, while the vector Oa_2 moves toward the axis of reference also through an angle of 30° to the new position Ob_2.

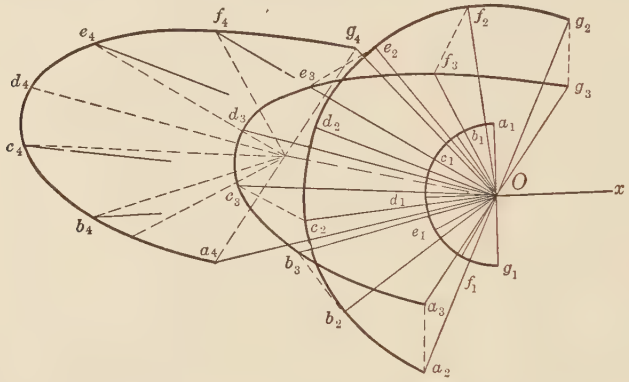

FIG. 35. — Graphical Determination of the Variation of $1/(VA)_r$ with θ_r. $Z_r = 1000$ Ohms; $S = 56.6$ Miles; No. 10 B. & S. Line

Their vector sum $Ob_1 + Ob_2 = Ob_3$. For $\theta_r = 0$, Oa_1 has moved to position Od_1 and Oa_2 to position Od_2, their vector sum being Od_3, and so on. When the limiting value of the phase angle $\theta_r = 90°$ is reached, Oa_1 has the final position Og_1, and Oa_2 the final position Og_2, their vector sum being Og_3. A curve drawn through the points a_3, b_3, c_3, $\cdot\ \cdot\ \cdot$, g_3, gives the locus of points of

$$Z_r \cosh^2 pS + \frac{Z_0^2}{Z_r} \sinh^2 pS,$$

as the angle θ_r varies from $-90°$ through 0° to $+90°$.

(2) To the same scale as the above two vectors of constant length and variable position, lay off the vector of fixed modulus and fixed argument $2 Z_0 \sinh pS \cosh pS$, making an angle with the

reference axis OX equal to $\beta_1 + \beta_2 + \alpha$. Adding this new vector vectorially to Oa_3, Ob_3, $Oc_3 \cdot \cdot \cdot$, Og_3, we find the vectors Oa_4, Ob_4, Oc_4, etc., which are the values of $1/(VA)_r$ for the assumed value of Z_r, and corresponding values of its variable phase angle θ_r. A curve drawn through the points a_4, b_4, c_4, etc., is the locus of points of $1/(VA)$ as θ_r varies between the limits $-90°$ and $+90°$.

(3) The actual receiver voltamperes may now be obtained by taking the reciprocal of $1/(VA)_r$. If we plot these values against θ_r, as shown in Fig. 36, we obtain a curve that represents the general law of variation of the voltamperes as a function of the phase angle of the receiver impedance.

An inspection of the graphical construction shows that the receiver voltamperes is a minimum when the three vectors are in

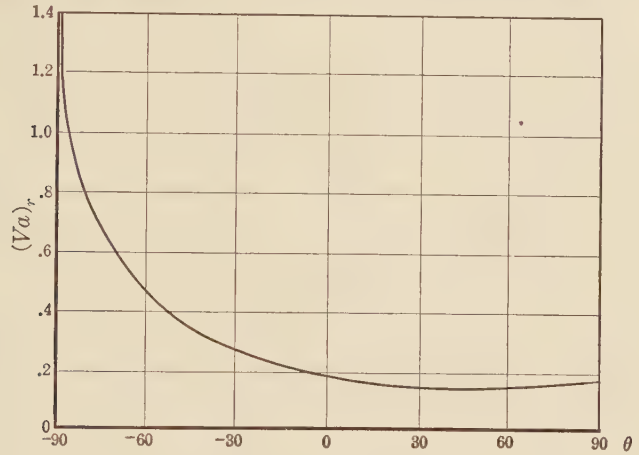

FIG. 36. — Variation of $(Va)_r$ with θ_r. $Z_r = 1000$; $S = \frac{1}{8}\lambda$; No. 10 B. & S. Line

phase with each other, for in that case $1/(VA)_r$ is a maximum. The arguments of the three vectors for this particular value of the voltamperes must be equal to each other, so that

$$2\beta_1 + \theta_r = 2\beta_2 + 2\alpha - \theta_r = \beta_1 + \beta_2 + \alpha,$$

whence

$$\theta_r = \beta_2 - \beta_1 + \alpha. \tag{23}$$

It will be observed that this angle is the argument of the quantity $Z_0 \tanh pS$. It follows, therefore, that while the receiver voltamperes is a maximum when the receiver impedance is equal to $Z_0 \tanh pS$ (see (13)), the voltamperes will be a minimum when the

phase angle of the receiver impedance is equal to the argument of $Z_0 \tanh pS$. For the 56.6-mile No. 10 B. & S. line, since $\beta_1 = \beta_2 = 90°$, it follows that the voltamperes will be a minimum when $\theta_r = +\alpha = -11.1°$, as shown in Fig. 37, when 50 volts at 796 cycles per second are applied at the sending end.

Figure 36 gives the variation of the voltamperes with the phase angle of the receiver impedance for a No. 10 B. & S. line whose length is one eighth of a wave-length, with an impressed voltage of 50 volts at a frequency of 796 cycles, and a receiver impedance of constant numerical value of 1000 ohms. The minimum voltamperes for this particular line under the above conditions occurs

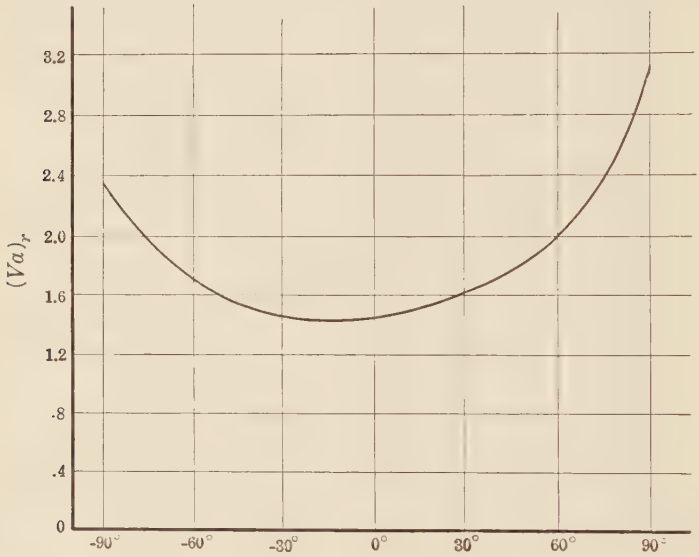

FIG. 37. — Variation of $(Va)_r$ with θ. $Z_r = 1000$; $S = \frac{1}{4}\lambda$

when the relative values of the receiver reactance and resistance are such that $\theta_r = 51.5°$.

79. Voltampere Transition and Reflection Coefficients. — Interesting and rather useful power relations may be obtained from a consideration of the voltage and current transition and reflection coefficients discussed in Chapter IX. We have, by (9) and (11) of that chapter,

$$E_r = K_{re}E_d, \text{ and } I_r = K_{ri}I_d.$$

Since $(VA)_r = E_r I_r$, we find

$$(VA)_r = K_{re} K_{ri} E_d I_d,$$

which may be written in the form

$$(VA)_r = K_{ret} E_d I_d,$$

where

$$K_{ret} = K_{re} K_{ri}$$

and where $E_d I_d$ represents the voltamperes at a point along an infinite line whose distance from the sending end is equal to the length of the line under consideration. Substituting for K_{re} and K_{ri} their values as given by (9) and (11), Chapter IX, we get

$$(24) \quad \frac{(VA)_r}{E_d I_d} = K_{ret} = \frac{4 Z_r Z_0}{\left[Z_r(1 + \epsilon^{-2pS}) + Z_0 (1 - \epsilon^{-2dS})\right]^2}.$$

In this expression all the terms are vectors. It is called the ***voltampere transition coefficient***, and may be defined as the ratio of the voltamperes in any terminal receiver to the voltamperes obtainable if the receiver impedance were equal to the characteristic line impedance in both modulus and argument.

From the discussion given in the preceding pages, it is evident that the voltampere transition coefficient may be larger than or smaller than unity, depending upon the length of the line, and upon the modulus and the phase angle of the receiver impedance.

The quantity $1 - K_{ret} = K_{\rho et}$ may be called ***voltampere reflection coefficient.*** It is a measure of the *terminal loss or gain* in voltamperes due to a difference between the receiver impedance and the characteristic line impedance, either in modulus or in argument or in both.

Assigning respective arguments to the vectors involved, we may write the expression (24) for the voltampere transition coefficient in the final form

$$(25) \quad K_{ret} = \frac{4 Z_r Z_0 \epsilon^{j(\theta_r + \alpha)}}{\left[Z_r \epsilon^{j\theta_r} (1 + \epsilon^{-2aS} \epsilon^{-j2vS}) + Z_0 \epsilon^{j\alpha} (1 + \epsilon^{-2aS} \epsilon^{-j2vS})\right]^2}.$$

To illustrate the application of the voltampere transition and reflection coefficients, consider a 42.3-mile No. 10 B. & S. line, with a sending-end voltage of 50 volts at a frequency of 796 cycles per second, feeding into a terminal receiving impedance of $300 \epsilon^{j60°}$ ohms. The voltampere transition coefficient for this particular case is

$$K_{ret} = \frac{4 \times 300 \times 723 \epsilon^{j(60° - 11.1°)}}{\left[300 (1 + \epsilon^{-2pS}) \epsilon^{j60°} + 723 (1 - \epsilon^{-2pS}) \epsilon^{-j11.1°}\right]^2}$$

where

$$\epsilon^{-2pS} = \epsilon^{-.774-j135^\circ} = .461\ (\cos 135^\circ - j \sin 135^\circ) = -.326 - j.326,$$

and

$$1 + \epsilon^{-2pS} = 1 - .326 - j.326 = .674 - j.326 = .75\ \epsilon^{j334.2^\circ},$$

$$1 - \epsilon^{-2pS} = 1 + .326 + j.326 = 1.326 + j.326 = 1.368\ \epsilon^{j13.8^\circ}.$$

Substituting these values in (25), and multiplying out, we get

$$K_{rei} = \frac{867600\ \epsilon^{j48.9^\circ}}{[225\ \epsilon^{j394.2^\circ} + 990\ \epsilon^{j2.7^\circ}]^2}$$

$$= \frac{867600\ \epsilon^{j48.9^\circ}}{1407500\ \epsilon^{j16^\circ}} = .617\ \epsilon^{j32.9^\circ}.$$

The voltampere reflection coefficient is

$$K_{pei} = 1 - K_{rei} = 1 - .617\ \epsilon^{j32.9^\circ} = .588\ \epsilon^{j34.7^\circ}.$$

It follows, therefore, that the voltamperes in the terminal impedance $300\ \epsilon^{j60^\circ}$ ohms is 61.7 per cent of the voltamperes obtainable with an end impedance equal to $Z_0 = 725\ \epsilon^{-j11.1^\circ}$, there being a reflection loss of 58.8 per cent.

The vector sum of the receiving-end voltamperes and reflected voltamperes will give the voltamperes for a receiving-end impedance equal to the characteristic line impedance in modulus and in argument.

80. Voltampere Transition and Reflection Coefficients when $Z_r = Z_0 \tanh pS$. — It was shown in § 77 that the receiving-end voltamperes for any value of θ_r will be a maximum when the receiving-end impedance is numerically equal to $Z_0 \tanh pS$. Assigning this value of Z_r in the expression for voltampere transition coefficient, we get

$$K_{rei} = \frac{4\ Z_0^2 \tanh pS\ \epsilon^{j(\theta_r+\alpha)}}{[Z_0 \tanh pS\ (1 + \epsilon^{-2pS})\epsilon^{j\theta_r} + Z_0\ (1 - \epsilon^{-2pS})\epsilon^{j\alpha}]^2}$$

$$(26) \qquad = \frac{4 \tanh pS\ \epsilon^{j(\theta_r+\alpha)}}{[\tanh pS\ (1 + \epsilon^{-2pS})\epsilon^{j\theta_r} + (1 - \epsilon^{-2pS})\epsilon^{j\alpha}]^2}.$$

For any definite value of θ_r, this expression will give the maximum obtainable voltamperes at the receiving end as a percentage of the voltamperes for $Z_r\epsilon^{j\theta_r} = Z_0\epsilon^{j\alpha}$.

Consider, for example, a line whose length is one-eighth of a wave-length or any odd multiple of that length, feeding into a

terminal impedance whose numerical value is equal to $Z_0 \tanh pS$ and whose phase angle $\theta_r = \alpha$. Since $\tanh pS = 1$ for such a line, it follows that

$$K_{ret} = \frac{4\,\epsilon^{j2\alpha}}{[(1+\epsilon^{-2pS})\epsilon^{j\alpha} + (1-\epsilon^{-2pS})\epsilon^{j\alpha}]^2},$$

which is seen by inspection to be equal to 1. It follows that for these particular lengths of line and for this value of the receiver impedance, there is no terminal gain or loss, as has been shown elsewhere. The results will be true approximately also for very long lines.

If, however, for the same length of line the phase angle of the receiver impedance is $\theta_r = \alpha$, the voltampere transition coefficient becomes

$$K_{ret} = \frac{4\,\epsilon^{j0^\circ}}{[(1+\epsilon^{-2pS})\epsilon^{-j\alpha} + (1-\epsilon^{-2pS})\epsilon^{j\alpha}]^2}.$$

For our illustrative No. 10 B. & S. line whose length is one-eighth of a wave-length (28.3 miles), we have

$$\epsilon^{-2pS} = .5969\,(\cos 90^\circ - j \sin 90^\circ) = -j.5969,$$

whence

$$(1+\epsilon^{-2pS})\epsilon^{-j\alpha} = (1 - j.5969)\,(\cos\alpha - j\sin\alpha),$$

$$(1-\epsilon^{-2pS})\epsilon^{j\alpha} = (1 + j.5969)\,(\cos\alpha + j\sin\alpha).$$

Adding these two equations, we have

$$(1+\epsilon^{-2pS})\,\epsilon^{-j\alpha} + (1-\epsilon^{-2pS})\,\epsilon^{j\alpha} = 2\,(\cos\alpha - .5969\sin\alpha).$$

Substituting in the equation for voltampere transition coefficient, we get

$$K_{ret} = \frac{4\,\epsilon^{j0^\circ}}{[2\,(\cos 11.1^\circ + .5969 \sin 11.1^\circ)]^2}$$

$$= \frac{1}{(.9813 + .115)^2} = .916$$

It follows from the above that the voltamperes will be 91.6 per cent of the voltamperes obtainable with $Z_r = Z_0 = 723\,\epsilon^{-j11.1^\circ}$, there being a terminal loss of 8.4 per cent.

By equations (13) and (23), the value of θ_r that would produce the minimum receiving-end voltamperes is the argument of the impedance whose modulus gives the maximum voltamperes, *i.e.*, the argument of the vector $Z_0 \tanh pS$, which is $\beta_2 - \beta_1 + \alpha$.

For the particular length of line under consideration (one-eighth of a wave-length), we have $\beta_2 - \beta_1 = 61.6°$ (see Table 2). Hence, the minimum value of receiving-end voltamperes when $Z_r = Z_0 \tanh pS$ occurs when $\theta_r = 61.6° - 11.1° = 50.5°$. Substituting this value in the expression for the transition coefficient, we get

$$K_{vet} = \frac{4\,\epsilon^{j(50.5°-11.1°)}}{[(1+\epsilon^{-2pS})\,\epsilon^{j50.5°} + (1-\epsilon^{-2pS})\epsilon^{-j11.1°}]^2},$$

where

$$1 + \epsilon^{-2pS} = 1 - j.5969 = 1.16\,\epsilon^{j329.2°},$$

$$1 - \epsilon^{-2pS} = 1 + j.5969 = 1.16\,\epsilon^{j30.8°}.$$

Hence we have

$$K_{vet} = \frac{4\,\epsilon^{j39.4°}}{[1.16\,\epsilon^{j379.7°} + 1.16\,\epsilon^{j19.7°}]^2}$$

$$= \frac{4\,\epsilon^{j39.4°}}{(2 \times 1.16\,\epsilon^{j19.7°})^2} = \frac{1}{1.16^2} = .745$$

The reflection coefficient is

$$K_{pet} = 1 - .745 = .255$$

The results obtained indicate that with the receiving-end impedance equal to the value for maximum receiving-end voltamperes, *i.e.*, $Z_r = Z_0 \tanh pS$, and with a phase angle $\theta_r = 50.5°$, the voltamperes is 74.5 per cent of the voltamperes obtainable if the receiving-end impedance were equal to the characteristic line impedance in modulus and in argument.

If the phase angle in the preceding case were $\theta_r = -50.5$, then by a calculation similar to the one given above, the voltampere transition coefficient would be

$$K_{vet} = 1.805,$$

and the reflection coefficient would be

$$K_{pet} = 1 - 1.805 = -.805$$

This indicates that the voltamperes is 180.5 per cent of the voltamperes obtainable with a receiving-end impedance equal to the characteristic line impedance in modulus and in argument, there being a terminal gain of 80.5 per cent.

81. Watts Transition and Loss Coefficients. — The useful component of the receiving-end voltamperes is the dissipative component, *i.e.*, the actual power in watts that operates the receiver.

If $(VA)_r$ denotes the receiving-end voltamperes, produced by a receiving-end impedance, $Z_r\epsilon^{j\theta r}$, and P_r denotes the watts dissipative component, we have

$$P_r = (VA)_r \cos \theta_r.$$

If E_s and I_s are the sending-end voltage and current, respectively, for the same line, either infinitely long or feeding a receiver impedance equal to the characteristic line impedance, then, by (24),

$$(VA)_r = K_{rei} E_s I_s \epsilon^{-2pS},$$

whence

$$P_r = K_{rei} E_s I_s \epsilon^{-2pS} \cos \theta_r,$$

or

$$P_r = K_{rei} \frac{E_s^2 \cos \theta_r}{Z_0} \epsilon^{-2pS}. \tag{27}$$

If the voltage $E_s\epsilon^{-pS}$ were impressed upon a receiver of impedance equal to the dissipative component, $R_0 = Z_0 \cos \alpha$, of the characteristic line impedance, the current would be in phase with the voltage, and we should have

$$I_r = \frac{E_r}{R_0} = \frac{E_s\epsilon^{-pS}}{Z_0 \cos \alpha}$$

whence

$$P_{rm} = \frac{E_s^2\epsilon^{-2pS}}{Z_0 \cos \alpha}, \tag{28}$$

where P_{rm} denotes the maximum dissipative power obtainable.

The ratio of the dissipative power in any receiver impedance $Z_r\epsilon^{j\theta r}$ to the maximum dissipative power obtainable is

$$\frac{P_r}{P_{rm}} = \frac{K_{rei} \frac{E_s^2}{Z_0} \epsilon^{-2pS} \cos \theta_r}{\frac{E_s^2 \epsilon^{-2pS}}{Z_0 \cos \alpha}} = K_{rei} \cos \theta_r \cos \alpha = K_{rw}. \tag{29}$$

This is called ***watts transition coefficient*** and is denoted by the symbol K_{rw}. Substituting for K_{rei} its value from (24), we obtain

$$\frac{P_r}{P_{rm}} = K_{rw} = \frac{4 Z_r Z_0 \epsilon^{j(\theta_r+\alpha)} \cos \theta_r \cos \alpha}{[Z_r (1 + \epsilon^{-2pS}) \epsilon^{j\theta_r} + Z_0 (1 - \epsilon^{-2pS}) \epsilon^{j\alpha}]^2}. \tag{30}$$

From the discussion just given, it follows that the *watts transition coefficient* gives the actual power consumed in the terminal receiver as a percentage of the maximum power obtainable at the end of the line. It is important to note that while the voltamperes transi-

tion coefficient K_{rei} may be greater than or smaller than unity, the watts transition coefficient never exceeds unity.

The quantity $1 - K_{rw} = K_{\rho w}$ may be called *watts loss coefficient* because it is a measure of the per cent loss in dissipative power in terms of the maximum power obtainable.

As an illustration of what precedes, consider a No. 10 B. & S. line whose length is one-eighth of a wave-length (28.2 miles). From what has been said, if the sending-end voltage is 50 volts at a frequency of 796 cycles per second, the maximum dissipative power is

$$P_{rm} = \frac{E_r^2 \epsilon^{-2pS}}{Z_0 \cos \alpha} = \frac{2500 \times .5969}{723 \times .9813} = 2.103 \text{ watts.}$$

If the receiver impedance were equal to the characteristic line impedance in both modulus and phase angle, *i.e.*, if $Z_r \epsilon^{j\theta_r} = 723\ \epsilon^{-j11.1°}$, the voltampere transition coefficient K_{rei} would be equal to 1, and the watt transition coefficient, by equation (29), would be

$$K_{rw} = \cos \theta_r \cos \alpha = \cos^2 \alpha = \cos^2 11.1° = .9628$$

The watt loss coefficient would be

$$K_{\rho w} = 1 - .9628 = .0372$$

It follows that the power consumed in the receiver impedance is 96.28 per cent of the maximum power obtainable, so that

$$P_r = .9628 \times 2.103 = 2.0247 \text{ watts,}$$

and the transition loss is 3.72 per cent, or

$$P_\rho = .0372 \times 2.103 = .0783 \text{ watts.}$$

For a 42.3 No. 10 B. & S. line feeding into an impedance of $Z_r \epsilon^{j\theta r} = 300\ \epsilon^{j60°}$, the watt transition coefficient would be

$$\begin{aligned} K_{rw} &= K_{rei} \cos 60° \cos 11.1° \\ &= .617\ \epsilon^{j32.9°} \times .500 \times .981 = .303\ \epsilon^{j32.9°}, \end{aligned}$$

the watt loss coefficient would be

$$\begin{aligned} K_{\rho w} &= 1 - .303\ \epsilon^{j32.9°} \\ &= 1 - .303\ (\cos 32.9° + j \sin 32.9°) = .746\ \epsilon^{-j12.5°}. \end{aligned}$$

Hence the power dissipated in the receiver impedance is 23.6 per cent of the maximum power obtainable, there being a transition loss of 76.4 per cent.

CHAPTER XI

THE EQUIVALENT CIRCUIT

82. Definition of the Equivalent Circuit. — By an equivalent circuit is meant a simple circuit which may replace a more complicated one, for analysis, test, or comparison. Such a circuit, with the same impressed voltage and frequency, will take the same current at the same power factor; in other words, it will absorb the same amount of energy as the circuit to which it is equivalent. Thus, in Fig. 38, circuit (*b*) may be considered as the equivalent of circuit (*a*), provided that

$$r_4 = r_1 + r_2 + r_3.$$

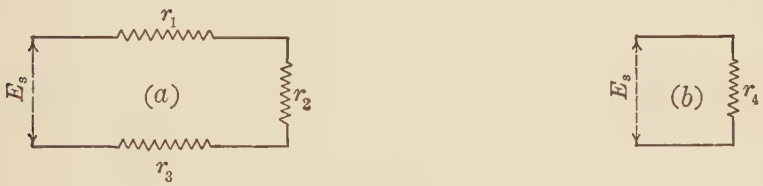

FIG. 38

Similarly, circuit (*d*), Fig. 39, may be thought of as the equivalent of circuit (*c*) if

$$r_3 = \frac{r_1 r_2}{r_1 + r_2}.$$

FIG. 39

83. Nominal and Equivalent Circuits of a Transmission Line. — A transmission line with uniformly distributed resistance, inductance, capacitance, and leakage conductance, though simple in

physical appearance, is, from an electrical point of view, a rather complex circuit of resistance and inductance elements in series, and capacitance, with leakage elements in parallel, as shown in Fig. 6, Chapter IV.

Such a line may be represented by an artificial circuit of at least three impedance branches in series-parallel arrangement. The artificial line, with the same impressed voltage at the same frequency, may *approximate*, more or less, line-end conditions; in this case, the artificial circuit is called a ***nominal circuit***.

On the other hand, the artificial line, with equal impressed voltages at the same frequency, may give *actual line-end conditions*. In such a case, the artificial line is called an ***equivalent circuit***.

In the *nominal circuit*, actual line resistance, inductance, capacitance, and leakage conductance are assumed lumped or concentrated, and the effect of their uniform distribution is therefore neglected, while in the *equivalent circuit*, the lumped resistance, inductance, capacitance, and leakage conductance differ numerically from, but are equal in effect to, the actual uniformly distributed line values.

An artificial circuit of a transmission line may be obtained by assuming that the total line capacitance and leakage conductance, which are normally distributed uniformly, are concentrated in the middle of the line. Evidently, one-half of the line-conductor resistance and inductance would be lumped on each side of the concentrated capacitance and leakage, as shown in Fig. 40.

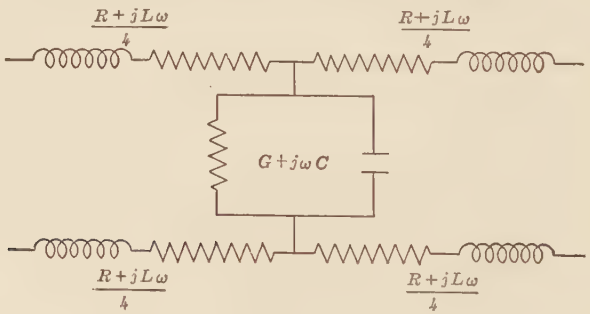

FIG. 40

Such an artificial two-wire circuit is called a *nominal H circuit*, or a *nominal I circuit*, from the similarity of the diagram with each of these two letters. If the values of the quantities in Fig. 40 are

such that the effect of the uniform distribution is taken care of, the artificial circuit would be an *equivalent H circuit*, or an *equivalent I circuit*.

The *H* circuit, nominal or equivalent, may be simplified by assuming the resistance and the inductive reactance that normally are in the two wires, as lumped into one single wire as shown in Fig. 41.

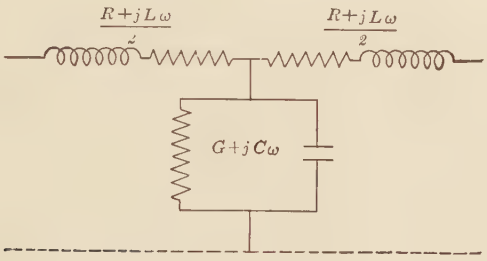

FIG. 41

The artificial circuit so obtained is known as the *nominal* ***T*** *circuit*, or *equivalent* ***T*** *circuit*, as the case may be, from the similarity of the diagram with the letter ***T***.

Another artificial circuit representing a transmission line may be built by splitting the total line capacitance and leakage conductance into two equal amounts, and lumping the line-wire resistance and inductance between the two halves, as shown in Fig. 42.

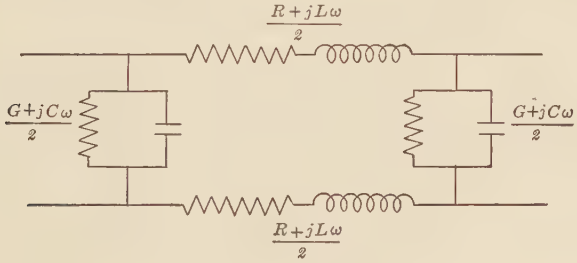

FIG. 42

This arrangement may be further simplified by lumping the total line resistance and inductive reactance into one single branch, resulting in the so-called ***nominal*** Π ***circuit***, if the effect of the uniform distribution is neglected, or the ***equivalent*** Π ***circuit***, if the

effect of the uniform distribution of line constants is taken care of. The diagram for this artificial circuit is shown in Fig. 43.

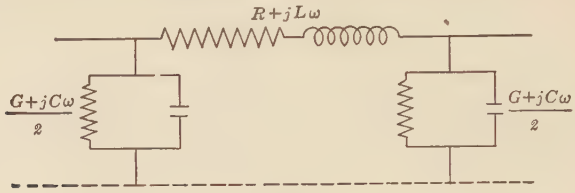

Fig. 43

84. The Nominal *T* Circuit. — Consider a line of length S, whose resistance, inductance, capacitance, and leakage conductance per mile of line are denoted by R, L, C, and G, respectively. If the frequency of the supply be known, the characteristic line impedance and the propagation constant, for that particular frequency are, respectively,

$$Z_0 = \sqrt{\frac{R + jL\omega}{G + jC\omega}}, \tag{1}$$

$$p = \sqrt{(R + jL\omega)\,(G + jC\omega)}. \tag{2}$$

Since the series line impedance per mile of line is $R + jL\omega$, its value for half line-length would be

$$\frac{(R + jL\omega)S}{2} = Z_{N_1} \tag{3}$$

where the subscript N indicates that Z_{N_1} refers to the nominal circuit. Since the parallel line admittance per mile of line is $G + jC\omega$, its equivalent impedance value for the entire length of line would be

$$\frac{1}{(G + jC\omega)\,S} = Z_{N_2}. \tag{4}$$

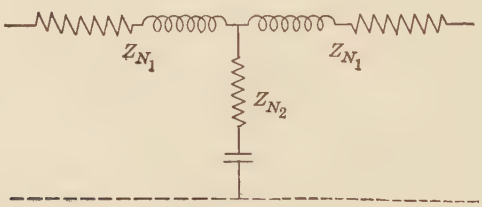

Fig. 44

Using this notation, the nominal *T* circuit of the line of length S is as shown diagrammatically in Fig. 44.

Instead of expressing the circuit branch impedances in terms of the line constants and frequency, it is more convenient to express them in terms of the characteristic impedance of the line and the propagation factor. Accordingly by combining equations (3) and (4) with equations (1) and (2), respectively, we obtain

$$Z_0 = \sqrt{2\,Z_{N_1}Z_{N_2}}, \tag{5}$$

$$pS = \sqrt{\frac{2\,Z_{N_1}}{Z_{N_2}}}, \tag{6}$$

from which we get

$$Z_{N_1} = \frac{Z_0 pS}{2}, \tag{7}$$

and

$$Z_{N_2} = \frac{Z_0}{pS}. \tag{8}$$

85. The Equivalent *T* Circuit. — Consider now the line of length S feeding into a terminal receiver whose impedance is equal to the characteristic line impedance in modulus and in argument. It was seen that such a line behaves in exactly the same manner as if the line were of infinite length. The diagram of the *T* circuit representing such a line is shown in Fig. 45.

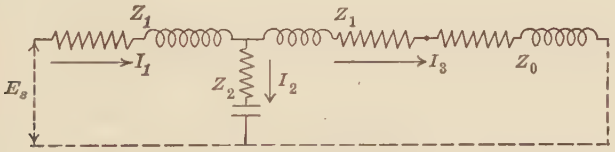

FIG. 45

Let I_1, I_2, and I_3 denote the currents in the respective branches, as indicated in the diagram. Keeping in mind that the quantities involved are vector quantities, and that the mathematical operations to be carried out are vector operations, the following relations may be obtained by an inspection of the diagram:

$$I_1 = I_2 + I_3, \tag{9}$$

$$I_2 Z_2 = I_3 (Z_1 + Z_0). \tag{10}$$

From (10) we have

$$I_2 = I_3 \frac{Z_1 + Z_0}{Z_2}. \tag{11}$$

Substituting this value in equation (9), we find

$$I_1 = I_3\left(1 + \frac{Z_1 + Z_0}{Z_2}\right),$$

or

$$I_1 = I_3 \frac{Z_0 + Z_1 + Z_2}{Z_2}, \tag{12}$$

whence

$$\frac{I_3}{I_1} = \frac{Z_2}{Z_0 + Z_1 + Z_2}. \tag{13}$$

From the nature of the line represented in Fig. 45, it is obvious that I_1 must be equal to the sending-end current I_s of the line, and that I_3, the load current in Z_0, must be equal to the current I_r at a point distant S from the sending end. Since we have in this case

$$I_r = I_s\epsilon^{-pS},$$

it follows that

$$I_3 = I_1\epsilon^{-pS},$$

or

$$\frac{I_3}{I_1} = \epsilon^{-pS}. \tag{14}$$

Equating the right sides of (13) and (14), we get

$$\epsilon^{-pS} = \frac{Z_2}{Z_0 + Z_1 + Z_2}. \tag{15}$$

This is a condition that must be satisfied if the circuit in Fig. 45 is to be fully equivalent to an infinite line, or to a line of any length loaded with a receiver impedance equal to the characteristic line impedance in modulus and in argument. Since the sending-end impedance, or the total impedance, of an infinitely long line is equal to the characteristic line impedance of the line, it follows that if the *T* circuit is to be the equivalent of the line, the combined impedance of the *T* circuit must be equal to Z_0.

It will be noticed that the circuit in Fig. 45 is built of an impedance branch $Z_1 + Z_0$, in parallel with a branch of impedance Z_2; and these two, so connected, are in series with a branch of impedance Z_1. By inspection, we see that the total impedance is

$$Z_T = Z_1 + \frac{Z_2\,(Z_1 + Z_0)}{Z_0 + Z_1 + Z_2} = Z_0. \tag{16}$$

Multiplying and collecting terms, we get

$$Z_0^2 = Z_1^2 + 2\,Z_1Z_2. \tag{17}$$

Equations (16) and (17) must be satisfied if true equivalence is to be obtained between the artificial T circuit and the line it represents, for these equations express the fundamental relations between the two circuits. The solutions of these two equations for Z_1 and Z_2 in terms of the characteristic impedance and the propagation factor will give the impedances that must be used in the branches of the T circuit. From (17) we have

$$Z_2 = \frac{Z_0^2 - Z_1^2}{2\,Z_1}. \tag{18}$$

Substituting in equation (15), and clearing of fractions, we get

$$\epsilon^{-pS} = \frac{Z_0^2 - Z_1^2}{2\,Z_1\left[(Z_0 + Z_1) + \dfrac{Z_0^2 - Z_1^2}{2\,Z_1}\right]} = \frac{Z_0^2 - Z_1^2}{(Z_0 + Z_1)^2},$$

or

$$\epsilon^{-pS} = \frac{Z_0 - Z_1}{Z_0 + Z_1} \tag{19}$$

and solving for Z_1, we get

$$Z_1 = Z_0 \frac{1 - \epsilon^{-pS}}{1 + \epsilon^{-pS}}. \tag{20}$$

Multiplying both numerator and denominator of the right side by $\epsilon^{pS/2}$, we get

$$Z_1 = Z_0 \frac{\epsilon^{pS/2} - \epsilon^{-pS/2}}{\epsilon^{pS/2} + \epsilon^{-pS/2}}.$$

Using the hyperbolic equivalent for the exponential functions, we may write

$$Z_1 = Z_0 \tanh \frac{pS}{2}, \tag{21}$$

where

$$\tanh \frac{pS}{2} = \sqrt{\frac{\cosh^2 \dfrac{aS}{2} - \cos^2 \dfrac{vS}{2}}{\cosh^2 \dfrac{aS}{2} - \sin^2 \dfrac{vS}{2}}}\, \epsilon^{j(\beta_3 - \beta_4)},$$

$$\beta_3 = \tan^{-1}\left(\tanh \frac{aS}{2} \tan \frac{vS}{2}\right),$$

$$\beta_4 = \tan^{-1} \frac{\tan \dfrac{vS}{2}}{\tanh \dfrac{aS}{2}}.$$

Hence the numerical value of the branch impedance Z_1 is

$$Z_1 = Z_0 \sqrt{\frac{\cosh^2 \frac{aS}{2} - \cos^2 \frac{vS}{2}}{\cosh^2 \frac{aS}{2} - \sin^2 \frac{vS}{2}}}, \tag{22}$$

and its argument is

$$\theta_1 = \beta_3 - \beta_4 + \alpha.$$

Substituting the value of Z_1 from (21) into equation (18), we get

$$Z_2 = Z_0 \frac{1 - \tanh^2 \frac{pS}{2}}{2 \tanh \frac{pS}{2}}. \tag{23}$$

But we know that

$$1 - \tanh^2 \frac{pS}{2} = 1 - \frac{\sinh^2 \frac{pS}{2}}{\cosh^2 \frac{pS}{2}}$$

$$= \frac{\cosh^2 \frac{pS}{2} - \sinh^2 \frac{pS}{2}}{\cosh^2 \frac{pS}{2}} = \frac{1}{\cosh^2 \frac{pS}{2}}.$$

Hence, by substitution in equation (23), we obtain

$$Z_2 = \frac{Z_0}{2 \cosh^2 \frac{pS}{2} \frac{\sinh \frac{pS}{2}}{\cosh \frac{pS}{2}}} = \frac{Z_0}{2 \sinh \frac{pS}{2} \cosh \frac{pS}{2}} = \frac{Z_0}{\sinh pS}. \tag{24}$$

Using the tensor notation, we get

$$Z_2 \epsilon^{j\theta_2} = \frac{Z_0 \epsilon^{j(\alpha - \beta_2)}}{\sqrt{\cos^2 aS - \cos^2 vS}}. \tag{25}$$

The expressions for Z_1 and Z_2, as given by equations (22) and (24), are the values of the impedances to be used in the respective branches of the ***T*** circuit, if it is to be a true equivalent of the line.

Accordingly, the line of length S may be represented by the artificial circuit shown in the diagram of Fig. 46.

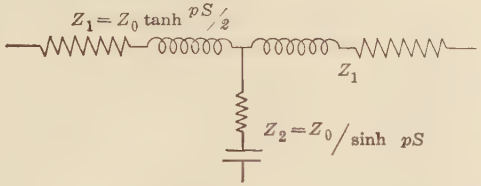

FIG. 46

86. Degree of Equivalence between the Nominal *T* Circuit and the Line it Represents. — The ratio between the branch impedances of the equivalent *T* circuit and the corresponding branch impedances of the nominal *T* circuit is a measure of the degree of equivalence obtainable between the nominal *T* circuit and the line it is supposed to represent. This ratio is

$$\frac{Z_1}{Z_{N_2}} = \frac{\tanh \frac{pS}{2}}{\frac{pS}{2}},$$

and it approaches unity only when the quantity pS approaches zero. Similarly, the ratio

$$\frac{Z_2}{Z_{N_2}} = \frac{pS}{\sinh pS}$$

approaches unity as pS approaches zero. Hence the degree of equivalence between the nominal *T* circuit and a line of definite propagation factor, *i.e.*, a line with definite electrical properties at a definite frequency, will depend upon the length of the line. The shorter the line, the closer the equivalence between it and its nominal *T* circuit. Consequently, the use of the nominal *T* must be restricted to short lines, with the understanding that even then the equivalence is only an approximate one.

87. Mathematical and Actual Equivalent *T* Circuits. — In many cases, the artificial *T* circuit, as well as the artificial Π circuit, cannot actually be constructed, because the resistance component of the calculated branch impedance Z_1 or Z_2 may be negative. Obviously, such an impedance has no physical significance. In

such a case, the artificial T (or Π) circuit is only the *mathematical equivalent* of the line it represents.

Such a case is illustrated in the following example, in which the equivalent T circuit of a 56.6-mile No. 10 B. & S. line is calculated under the assumption that the frequency of the supply is 796 cycles per second. From Table 2, we have

$$\tanh \frac{pS}{2} = 1.00\,\epsilon^{j61.6°}, \quad \text{and} \quad \sinh pS = 1.132\,\epsilon^{j90°}.$$

Hence

$$Z_1\epsilon^{j\theta_1} = Z_0 \tanh \frac{pS}{2} = 723\,\epsilon^{-j11.1°} \times \epsilon^{j61.6°} = 723\,\epsilon^{j50.5°},$$

$$R_1 = Z_1 \cos\theta_1 = 723 \cos 50.5° = 461 \quad \text{ohms resistance,}$$

$$X_1 = Z_1 \sin\theta_1 = 723 \sin 50.5° = 560 \quad \text{ohms reactance.}$$

Since $X_1 = L_1\omega$, it follows that

$$L_1 = \frac{560}{5000} = .112 \text{ henries.}$$

Similarly,

$$Z_2\epsilon^{j\theta_2} = \frac{Z_0}{\sinh pS} = \frac{723\,\epsilon^{-j11.1°}}{1.132\,\epsilon^{j90°}} = 637.5°\epsilon^{-j101.1°},$$

$$R_2 = Z_2 \cos\theta_2 = 637.5 \cos(-101.1°) = -122.5 \quad \text{ohms resistance,}$$

$$X_2 = Z_2 \sin\theta_2 = 637.5 \sin(-101.5°) = -626 \quad \text{ohms reactance.}$$

Since X_2 is negative, it follows that it is a dielectric reactance

$$X_2 = \frac{1}{C_2\omega} = 626 \quad \text{ohms reactance,}$$

whence

$$C_2 = \frac{1}{526 \times 5000} = .3208 \times 10^{-6} \quad \text{farads.}$$

Since the resistance component of the branch impedance Z_2 is a negative quantity, it has no physical meaning, and the artificial equivalent T circuit, with branch impedances of values as calculated above, cannot be actually constructed. It is therefore only the mathematical equivalent of the line.

The branch impedances of the nominal T circuit for this line are

$$Z_{N_1} = \frac{Z_0\,pS}{2} = 723\,\epsilon^{-j11.1°} \times \frac{.0292\,\epsilon^{j71.8°} \times 56.6}{2} = 597\,\epsilon^{j60.7°},$$

$$Z_{N_2} = \frac{Z_0}{pS} = \frac{723\,\epsilon^{-j11.1°}}{.0292\,\epsilon^{j71.8°} \times 56.6} = 437.3\,\epsilon^{-j82.9°}.$$

The degree of equivalence between the nominal T circuit and the equivalent T circuit, *i.e.*, between the nominal T circuit and the line itself, for the Z_{N_1} branch, is

$$\frac{Z_{N_1}}{Z_1} = \frac{597.6\ \epsilon^{j60.7°}}{723\ \epsilon^{j50.5°}} = .826\ \epsilon^{j10.2°},$$

and, for the Z_{N_2} branch,

$$\frac{Z_{N_2}}{Z_2} = \frac{437.3\ \epsilon^{-j82.9°}}{637.5\ \epsilon^{-j101.1°}} = .686\ \epsilon^{j18.2°}.$$

It will be observed, that while the nominal T circuit may actually be constructed, Z_{N_1} is only 82.6 per cent, and Z_{N_2} is 68.6 per cent of the numerical values of the equivalent branch impedances, and their phase angles differ by 10.2° and 18.2°, respectively, from true equivalence with the line they are supposed to represent. The use of the nominal T circuit in the analysis of the behavior of

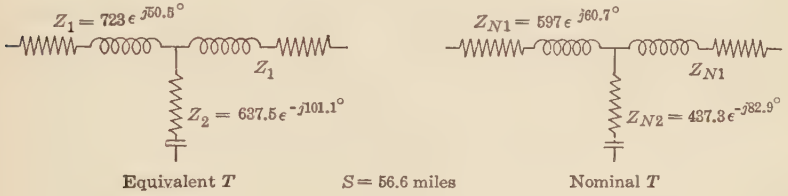

FIG. 47

this line will therefore lead to erroneous results. Fig. 47 gives the diagrams of the equivalent and nominal T circuits of the 56.6-mile No. 10 B. & S. line.

The equivalent T circuit of a line half as long as the one considered above, gives the following values for the respective branch impedances:

$$\tanh \frac{pS}{2} = .431\ \epsilon^{j69.7°},$$

$$\sinh pS = .736\ \epsilon^{j75.8°}.$$

Hence

$$Z_1\epsilon^{j\theta_1} = 723\ \epsilon^{-j11.1°} \times .431\ \epsilon^{j69.7} = 311\ \epsilon^{j58.6°},$$

$$R_1 = 311.6 \cos 58.6° = 162 \quad \text{ohms resistance},$$

$$X_1 = 311.6 \sin 58.6° = 258.8 \quad \text{ohms reactance},$$

$$L_1 = \frac{265.8}{5000} = .05316 \quad \text{henries inductance},$$

$$Z_2\epsilon^{j\theta_2} = \frac{723\ \epsilon^{-j11.1°}}{.736\ \epsilon^{j75.8°}} = 983\ \epsilon^{-j86.9°},$$

$$R_2 = 983 \cos (-86.9°) = 50.18 \quad \text{ohms resistance,}$$
$$X_2 = 983 \sin (-86.9°) = -981 \quad \text{ohms capacity reactance,}$$
$$C_1 = \frac{1}{981 \times 5000} = .2 \times 10^{-6} \quad \text{farads.}$$

The results obtained above show that the equivalent T circuit of a No. 10 B. & S. line 28.3 miles long may actually be built.

The impedance branches of the nominal T circuit of this line have the following values:

$$Z_{N_1} = 723\, \epsilon^{-j11.1°} \times \frac{.0292\, \epsilon^{j71.8°} \times 28.2}{2} = 297.7\, \epsilon^{j60.7°},$$
$$Z_{N_2} = \frac{723\, \epsilon^{-j11.1°}}{.0292\, \epsilon^{j71.8°} \times 28.2} = 877\, \epsilon^{-j82.9°}.$$

The degree of equivalence between the nominal and the equivalent T circuit, *i.e.*, between the nominal T circuit and the line itself, is, for branch Z_{N_1},

$$\frac{Z_{N_1}}{Z_1} = \frac{297.7\, \epsilon^{j60.7°}}{311\, \epsilon^{j58.6°}} = .957\, \epsilon^{j2.1°},$$

and, for branch Z_{N_2},

$$\frac{Z_{N_2}}{Z_1} = \frac{877\, \epsilon^{-j82.9°}}{983\, \epsilon^{-j86.9°}} = .882\, \epsilon^{j4°}.$$

It will be observed that the degree of equivalence between the nominal T circuit and the line of this particular length is very much closer than in the previous case, both numerically and in phase.

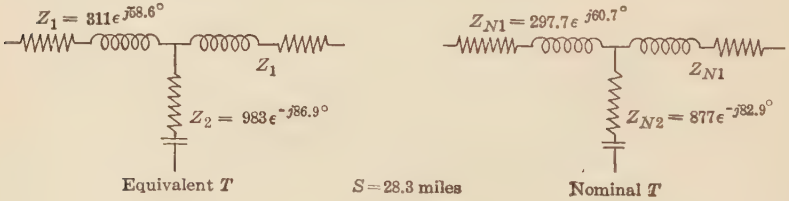

FIG. 48

88. Equivalent T Circuits for Lines with Definite End Conditions. The Open-Circuited Line. — The values of the branch impedances Z_1 and Z_2 of the equivalent T circuit, as given by equations (21) and (24), respectively, hold for all terminal conditions of the line the artificial circuit is supposed to represent. This may be proved from the fact that the hyperbolic formulas for the open-cir-

cuited line (Chapter VI), for the short-circuited line (Chapter VII) and for the end-loaded line (Chapter VIII) may easily be obtained from an analysis of their artificial *T* equivalent circuits. The analysis will also show that, with branch impedances known, line calculations may be made with comparative facility by the use of the equivalent *T* circuit.

Consider the equivalent *T* circuit, Fig. 49, of a line of length S miles. Assuming the line to be open-circuited at its end, it is obvious that in its equivalent *T* circuit, branch D has no effect whatever. Hence the combined impedance of the remaining two branches A and B must be equal to the sending-end impedance of the open-circuited line, that is,

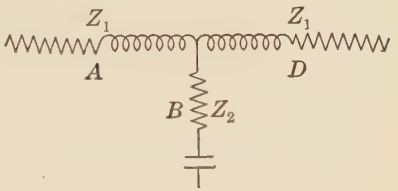

FIG. 49

$$Z_s = Z_1 + Z_2, \tag{26}$$

where the addition must be a vector addition. Substituting for Z_1 and Z_2 their values in terms of the characteristic impedance and the propagation factor of the line from (21) and (24), we find

$$Z_s = Z_0\left[\tanh\frac{pS}{2} + \frac{1}{\sinh pS}\right]$$

$$= Z_0\left(\frac{\cosh pS - 1}{\sinh pS} + \frac{1}{\sinh pS}\right) = \frac{Z_0}{\tanh pS},$$

which is equation (2), Chapter VI.

The sending-end current (see (8), Chapter VI) will be,

$$I_s = \frac{E_s}{Z_s} = \frac{E_s}{Z_0}\tanh pS. \tag{27}$$

The voltage at the open-circuited end of the line (see (15), Chapter VI) is

$$E_r = I_s Z_2 = E_s\frac{Z_2}{Z_1 + Z_2} = \frac{E_s}{\cosh pS}. \tag{28}$$

As an illustration of the use of the equivalent circuit, consider the *T* equivalent of the 56.6-mile No. 10 B. & S. line open-circuited at its end. Using the branch impedance values obtained in the preceding section, we have

$$Z_s = Z_1 + Z_2 = 723\,\epsilon^{j30.5^\circ} + 637.5\,\epsilon^{-j101.1^\circ}$$

$$= 338.5 - 66\,j = 344\,\epsilon^{-j11.1^\circ} \quad \text{ohms.}$$

Similarly, the voltage at the open-circuited end, by (28), is

$$E_r = E_s \frac{637.5\ \epsilon^{-j101.1^\circ}}{723\ \epsilon^{j50.5^\circ} + 637.5\ \epsilon^{-j101.4^\circ}}.$$

If 50 volts at a frequency of 796 cycles per second is impressed at the sending end, this becomes

$$E_r = \frac{31875\ \epsilon^{-j101.1^\circ}}{344\ \epsilon^{-j11.1^\circ}} = 92.7\ \epsilon^{-j90^\circ} \quad \text{volts.}$$

89. The Equivalent *T* Circuit of a Short-Circuited Line. — Figure 50 shows diagrammatically the equivalent *T* circuit of a line short-circuited at its end. The sending-end impedance is equal to

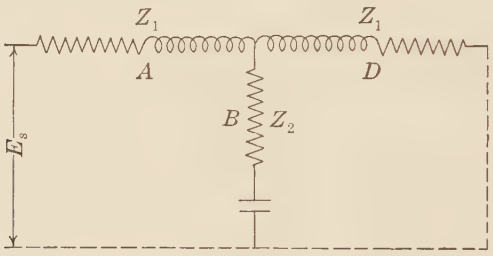

FIG. 50

the combined impedance of the *T* circuit, which, by inspection (see (3), Chapter VII) is seen to be

$$(29) \qquad Z_s = \frac{1}{\frac{1}{Z_1} + \frac{1}{Z_2}} + Z_1 = \frac{Z_1^2 + 2\,Z_1 Z_2}{Z_1 + Z_2}$$

$$= Z_0 \tanh pS.$$

Hence, the sending-end current (see (4), Chapter VII) would be

$$(30) \qquad I_s = \frac{E_s}{Z_s} = \frac{E_s(Z_1 + Z_2)}{Z_1^2 + 2\,Z_1 Z_2} = \frac{E_s}{Z_0 \tanh pS}.$$

The current at the short-circuited end may be found as follows. Since $I_s = I_A$, and $I_r = I_D$, it will be seen that $I_s = I_B + I_D$, and $I_B Z_2 = I_D Z_1$. Hence

$$I_B = I_D \frac{Z_1}{Z_2},$$

and

$$I_s = I_D \left[1 + \frac{Z_1}{Z_2}\right] = I_D \frac{Z_1 + Z_2}{Z_2},$$

whence

$$(31) \qquad I_D = I_r = I_s \frac{Z_2}{Z_1 + Z_2} = \frac{I_s}{\cosh pS}.$$

To illustrate the application of this theory, consider the 56.6-mile No. 10 B. & S. line short-circuited at its end. By equation (29), the sending-end impedance is

$$Z_s = \frac{723^2 \epsilon^{j2\times 50.5^\circ} + 2 \times 723 \epsilon^{j50.5^\circ} \times 637.5 \epsilon^{-j101.1^\circ}}{723 \epsilon^{j50.5^\circ} + 637.5 \epsilon^{-j101.1^\circ}}$$
$$= 1525 \epsilon^{-j11.1^\circ} \text{ ohms impedance.}$$

The currents at the sending end and at the short-circuited end of the line may be calculated in a similar manner, by the use of equations (30) and (31), respectively.

90. The Equivalent *T* Circuit of a Line with any End Impedance $Z_r\epsilon^{j\theta_r}$.—The equivalent *T* circuit of a line with a terminal impedance

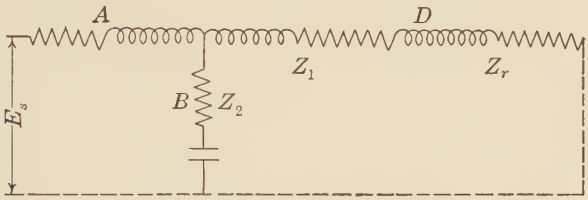

FIG. 51

is shown in Fig. 51. As in the cases previously discussed, the sending-end impedance is equal to the combined impedance of the *T* circuit. By inspection, this is seen to be

$$(32) \quad Z_s = \frac{1}{\frac{1}{Z_1 + Z_r} + \frac{1}{Z_2}} + Z_1 = \frac{Z_1^2 + 2\,Z_1Z_2 + Z_r(Z_1 + Z_2)}{Z_1 + Z_2 + Z_r}.$$

That this is equivalent to the sending-end impedance of a line loaded with a terminal impedance Z_r, as given by equation (5), Chapter VIII, may be proven as follows. Using the values of Z_1 and Z_2 as given by equations (21) and (24), we have

$$Z_1^2 = Z_0^2 \tanh^2 \frac{pS}{2} = Z_0^2 \frac{(\cosh pS - 1)^2}{\sinh^2 pS},$$

$$2\,Z_1Z_2 = Z_0^2 \left[2 \tanh \frac{pS}{2} \times \frac{1}{\sinh pS} \right]$$
$$= 2\,Z_0^2 \frac{(\cosh pS - 1)}{\sinh^2 pS}.$$

Hence

$$2\,Z_1Z_2 + Z_1^2 = Z_0^2\frac{(\cosh pS - 1)^2 + 2\,(\cosh pS - 1)}{\sinh^2 pS}$$

$$= Z_0^2\frac{\cosh^2 pS + 1 - 2\cosh pS + 2\cosh pS - 2}{\sinh^2 pS}$$

$$= Z_0^2\frac{\cosh^2 pS - 1}{\sinh^2 pS} = Z_0^2.$$

Similarly,

$$Z_1 + Z_2 = Z_0\left[\tanh\frac{pS}{2} + \frac{1}{\sinh pS}\right] = Z_0\frac{1}{\tanh pS},$$

whence, by substitution,

$$Z_s = \frac{Z_0^2 + Z_0Z_r\,\mathrm{ctnh}\,pS}{Z_0\,\mathrm{ctnh}\,pS + Z_r},$$

or

$$Z_s = \frac{Z_0^2 + Z_0Z_r\dfrac{\cosh pS}{\sinh pS}}{Z_0\dfrac{\cosh pS}{\sinh pS} + Z_r} = \frac{Z_0^2\sinh pS + Z_0Z_r\cosh pS}{Z_0\cosh pS + Z_r\sinh pS},$$

which is equation (5), Chapter VIII.

Expressions for the sending-end current and the receiving-end current may be obtained from the T circuit, as follows. Since $I_s = I_A$ and $I_r = I_D$, and since $I_BZ_2 = I_r\,(Z_1 + Z_r)$, we find

$$I_B = I_r\frac{Z_1 + Z_r}{Z_2},$$

and

$$I_s = I_B + I_r.$$

Substituting in this last expression the value of I_B given above, we get

$$I_s = I_r\frac{Z_1 + Z_2 + Z_r}{Z_2},$$

or

$$I_r = I_s\frac{Z_2}{Z_1 + Z_2 + Z_r}.$$

Since $I_s = E_s/Z_s$, if we substitute the value of Z_s from (32), we get

$$(33)\qquad I_s = E_s\frac{Z_1 + Z_2 + Z_r}{Z_1^2 + 2\,Z_1Z_2 + Z_r(Z_1 + Z_2)}.$$

Substituting this value of I_s in the above expression for the receiving-end current, we obtain

$$(34)\qquad I_r = E_s\frac{Z_2}{Z_1^2 + 2\,Z_1Z_2 + Z_r(Z_1 + Z_2)}.$$

As an illustration, suppose we are required to calculate the receiving-end current in a load impedance of $600\,\epsilon^{j45°}$ ohms, connected to a 56.6-mile No. 10 B. & S. line, under the assumption that at the sending end the impressed electromotive force is 50 volts at a frequency of 796 cycles. From previous numerical illustrations for a line of this length, we have

$$Z_1\epsilon^{j\theta_1} = 723\,\epsilon^{j50.5°}, \quad Z_2\epsilon^{j\theta_2} = 637.5\,\epsilon^{-j101.1°}.$$

Hence, by (34),

$$I_r\epsilon^{j\theta_3} = \frac{50 \times 637.5\,\epsilon^{-j101.1°}}{723^2\epsilon^{j101°} + 2\times 723\epsilon^{j50.5°}\times 637.5\epsilon^{-j101.1°} + 600\epsilon^{j45°}(723\epsilon^{j50.5°} + 637.5\epsilon^{-j101.1°})}$$
$$= .048\,\epsilon^{-j93.7°}$$

The angle 93.7° is the lagging angle of the receiver current behind the sending-end voltage.

The terminal receiver voltage is

$$E_r = I_rZ_r = .048\,\epsilon^{-j93.7°} \times 600\,\epsilon^{j45°} = 28.8\,\epsilon^{-j48.7°} \text{ volts.}$$

It lags behind the sending-end voltage by an angle of 48.7°. Obviously, the phase angle between the receiver current and receiver voltage is determined by the phase angle of the receiver impedance, which in this particular case is $-48.7° - (-93.7°) = 45°$.

91. The Nominal Π Circuit. — Consider a line of length S having a propagation constant p and characteristic impedance Z_0. The artificial nominal Π circuit, in which actual line values are supposed to be concentrated in the branch impedances is shown in Fig. 52. Using the same method of analysis as for the nominal ***T*** circuit, we have

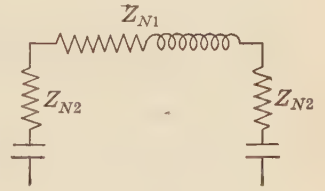

FIG. 52

$$(35) \qquad Z_{N_1} = (R + jL\omega)\,S,$$

$$(36) \qquad Z_{N_2} = \frac{2}{(G + jC\omega)\,S}.$$

Combining these two equations with the known expressions for Z_0 and p, we get

$$(37) \qquad Z_0 = \sqrt{\frac{Z_{N_1}\,Z_{N_2}}{2}},$$

and

$$(38) \qquad pS = \sqrt{\frac{2\,Z_{N_1}}{Z_{N_2}}}.$$

Solving these two equations for Z_{N_1} and Z_{N_2}, we obtain the values of the branch impedances of the nominal Π circuit in terms of the characteristic impedance and propagation factor of the line it represents

$$Z_{N_1} = Z_0 pS, \tag{39}$$

$$Z_{N_2} = \frac{2\,Z_0}{pS}. \tag{40}$$

It is of interest to note that the values of the branch impedances for the nominal Π circuit are twice the values of the corresponding branch impedances of the nominal ***T*** circuit.

92. The Equivalent Π Circuit. — Consider a line of length S feeding into an impedance equal to the characteristic line impedance in modulus and in argument. Such a line, since it behaves like an

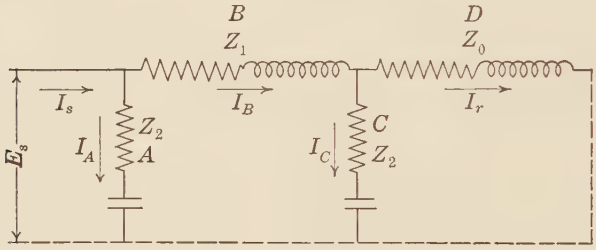

FIG. 53

infinite line, will have a sending-end impedance equal to the characteristic line impedance Z_0. The equivalent Π circuit of such a line is shown in Fig. 53.

Keeping in mind that the quantities involved are vectors, and that the sending-end impedance of the line must be equal to the combined impedance of its equivalent Π circuit, we have, by inspection of Fig. 53,

$$Z_\pi = Z_0 = Z_s = \frac{1}{\frac{1}{Z_2} + \frac{Z_0 + Z_2}{Z_0Z_1 + Z_1Z_2 + Z_0Z_2}}.$$

Clearing of fractions and solving for Z_0, we find

$$Z_0 = Z_2\sqrt{\frac{Z_1}{Z_1 + 2\,Z_2}}. \tag{41}$$

By an inspection of Fig. 53 relative to the currents in the respective branches, the following relations may also be obtained

$$I_s = I_A + I_B, \tag{42}$$

$$I_B = I_C + I_r, \tag{43}$$

$$I_A Z_2 = I_B Z_1 + I_C Z_2, \tag{44}$$

$$I_C Z_2 = I_r Z_0. \tag{45}$$

From the last equation, we get

$$I_C = I_r \frac{Z_0}{Z_2}. \tag{46}$$

Substituting in equation (43), we find

$$I_B = I_r \left(\frac{Z_0}{Z_2} + 1\right) = I_r \frac{Z_0 + Z_2}{Z_2}. \tag{47}$$

Substituting this value of I_B in equation (42), we get

$$I_s = I_r \frac{Z_0 + Z_2}{Z_2} + I_A. \tag{48}$$

From (44), (46), (47), we have

$$I_A Z_2 = I_r Z_1 \frac{Z_0 + Z_2}{Z_2} + I_r Z_2 \frac{Z_0}{Z_2},$$

whence

$$I_A = I_r \left[\frac{Z_0 Z_1 + Z_1 Z_2}{Z_2^2} + \frac{Z_0}{Z_2}\right]. \tag{49}$$

Substituting in (48), we find

$$I_s = I_r \left[\frac{Z_0 + Z_2}{Z_2} + \frac{Z_0 Z_1 + Z_1 Z_2}{Z_2^2} + \frac{Z_0}{Z_2}\right],$$

or

$$I_s = I_r \frac{Z_2^2 + 2\,Z_0 Z_2 + Z_0 Z_1 + Z_1 Z_2}{Z_2^2}, \tag{50}$$

whence

$$\frac{I_r}{I_s} = \frac{Z_2^2}{Z_2^2 + 2\,Z_0 Z_2 + Z_0 Z_1 + Z_1 Z_2} = \epsilon^{-pS}, \tag{51}$$

by virtue of the fact that the circuit is equivalent to an infinite line.

Equations (41) and (51) express the necessary conditions that must be satisfied if the artificial Π circuit is to be fully the equivalent with respect to end values, of the line of length S with the characteristic impedance Z_0 and with a propagation constant p. The solutions of these two equations for Z_1 and Z_2 will, therefore,

give the values of the branch impedances that must be used in the equivalent Π circuit.

From equation (41), by solving for Z_1, we get

$$Z_1 = \frac{2 Z_0^2 Z_2}{Z_2^2 - Z_0^2}. \tag{52}$$

Substituting in equation (51), we have

$$\epsilon^{-pS} = \frac{Z_2^2}{Z_2^2 + 2 Z_0 Z_2 + (Z_0 + Z_2) \dfrac{2 Z_0^2 Z}{Z_2^2 - Z_0^2}},$$

which reduces to

$$\epsilon^{-pS} = \frac{Z_2 - Z_0}{Z_2 + Z_0},$$

or

$$Z_2 (1 - \epsilon^{-pS}) = Z_0 (\epsilon^{-pS} + 1).$$

Hence

$$Z_2 = Z_0 \frac{1 + \epsilon^{-pS}}{1 - \epsilon^{-pS}}. \tag{53}$$

Multiplying both numerator and denominator by $\epsilon^{pS/2}$, we get

$$Z_2 = Z_0 \frac{\epsilon^{pS/2} + \epsilon^{-pS/2}}{\epsilon^{pS/2} - \epsilon^{-pS/2}}. \tag{54}$$

Substituting the hyperbolic equivalent for the exponentials, we find

$$Z_2 = \frac{Z_0}{\tanh \dfrac{pS}{2}} = Z_0 \operatorname{ctnh} \frac{pS}{2}. \tag{55}$$

To obtain the value of Z_1, substitute this value of Z_2 in (52). This gives

$$Z_1 = \frac{2 Z_0^3 \operatorname{ctnh} \dfrac{pS}{2}}{Z_0^2 \operatorname{ctnh}^2 \dfrac{pS}{2} - Z_0^2} = 2 Z_0 \frac{\operatorname{ctnh} \dfrac{pS}{2}}{\operatorname{ctnh}^2 \dfrac{pS}{2} - 1}$$

$$= 2 Z_0 \frac{\cosh \dfrac{pS}{2}}{\sinh \dfrac{pS}{2}} \times \frac{\sinh^2 \dfrac{pS}{2}}{\cosh^2 \dfrac{pS}{2} - \sinh^2 \dfrac{pS}{2}}$$

$$= 2 Z_0 \cosh \frac{pS}{2} \sinh \frac{pS}{2},$$

or, finally,

$$Z_1 = Z_0 \sinh pS. \tag{56}$$

Equations (55) and (56) give the values of the branch impedances of the equivalent Π circuit in terms of the characteristic impedance of the line and its propagation factor pS.

93. Degree of Equivalence between the Nominal Π Circuit and the Line it is Supposed to Represent. — The ratio of the branch impedances of the equivalent Π circuit to the respective branch impedances of the nominal Π circuit, is a measure of the degree of equivalence between the nominal Π circuit and the line itself. This ratio is

$$\frac{Z_1}{Z_{N_1}} = \frac{\sinh pS}{pS}, \tag{57}$$

which is seen to approach unity as S approaches zero. Similarly,

$$\frac{Z_2}{Z_{N_2}} = \frac{\frac{pS}{2}}{\tanh \frac{pS}{2}}, \tag{58}$$

which also approaches unity as S approaches zero. Hence it follows that for lines with definite electrical characteristics, the degree of equivalence between the nominal Π circuit and he line will depend upon the length of the line; the shorter the line, the closer the degree of equivalence. As in the case of nominal Π circuit, the use of the nominal Π circuit must be restricted to very short lines, with the understanding that even then the equivalence is only approximate.

As an illustration, the equivalent Π circuit of the 56.6-mile No. 10 B. & S. line is calculated as follows.

$$Z_1\epsilon^{j\theta_1} = Z_0 \sinh pS = 723\ \epsilon^{-j11.1^\circ} \times 1.132\ \epsilon^{j90^\circ} = 820\ \epsilon^{j78.9^\circ},$$

$$Z_2\epsilon^{j\theta_2} = \frac{Z_0}{\tanh \frac{pS}{2}} = \frac{723\ \epsilon^{-j11.1^\circ}}{1\ \epsilon^{j61.6^\circ}} = 723\ \epsilon^{-j72.7^\circ}.$$

Hence

$R_1 = Z_1 \cos\theta_1 = 820 \cos 78.9^\circ = 157.5$ ohms resistance,

$X_1 = Z_1 \sin\theta_1 = 820 \sin 78.9^\circ = 815$ ohms inductive reactance,

$L_1 = \frac{815}{5000} = .163$ henries,

$Z_2 = \frac{723\ \epsilon^{-j11.1^\circ}}{\epsilon^{j61.6^\circ}} = 723\ \epsilon^{-j72.7^\circ}$ ohms impedance,

$R_2 = Z_1 \cos\theta_1 = 723 \cos(-72.7^\circ) = 215$ ohms resistance,

$X_2 = Z_1 \sin \theta_1 = 723 \sin (-72.7°) = -690$ ohms capacity reactance,

$$C_2 = \frac{1}{5000 \times 690} = .29 \text{ micro-farads.}$$

It follows that the equivalent Π circuit of this particular line may be built as shown in Fig. 54. It will be observed that while the

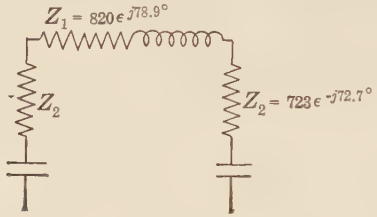

Fig. 54

equivalent *T* circuit of this particular length of line is only its mathematical equivalent, the equivalent Π circuit is not only its mathematical equivalent but al o its physical equivalent, for it may actually be constructed.

94. The Equivalent Π Circuit of an Open-Circuited Line. — Figure 55 shows the equivalent Π circuit of an open-circuited line.

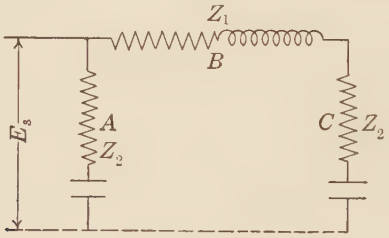

Fig. 55

Obviously, the sending-end impedance of the line must be equal to the combined impedance of the equivalent circuit, which, by inspection, is

$$Z_\pi = Z_s = \frac{1}{\frac{1}{Z_1 + Z_2} + \frac{1}{Z_2}},$$

whence

$$Z_s = \frac{Z_1 Z_2 + Z_2^2}{Z_1 + 2\,Z_2} = \frac{Z_0}{\tanh pS}. \tag{59}$$

This is equivalent to (2), Chapter VI.

For the 56.6-mile No. 10 B. & S. line, we have

$$Z_s = \frac{820\,\epsilon^{j78.9^\circ} \times 723\,\epsilon^{-j72.7^\circ} + 723^2\epsilon^{-j145.4^\circ}}{820\,\epsilon^{j78.9^\circ} + 2 \times 723\,\epsilon^{-j72.7^\circ}}$$

$$= 344\,\epsilon^{-j11.1^\circ} \text{ ohms impedance.}$$

The sending-end current is

$$I_s = \frac{E_s}{Z_s} = E_s \frac{Z_1 + 2Z_2}{Z_1 Z_2 + Z_2^2}. \tag{60}$$

The voltage at the open-circuited end, by referring to Fig. 55, is

$$E_r = I_C Z_2$$

and since $I_C = I_s - I_A$ and $I_A Z_2 = E_s$, it follows that

$$E_r = (I_s - I_A)Z_2 = \left(I_s - \frac{E_s}{Z_2}\right)Z_2 = I_s Z_2 - E_s.$$

Substituting for the value of the sending-end current from (60), we get

$$E_r = E_s\left[\frac{Z_1 Z_2 + 2Z_2^2}{Z_1 Z_2 + Z_2^2} - 1\right]$$

or

$$E_r = E_s \frac{Z_2}{Z_1 + Z_2} = \frac{E_s}{\cosh pS}. \tag{61}$$

95. Equivalent Π Circuit of a Short-Circuited Line. — Figure 56 shows the diagram of the equivalent Π circuit of a line with the receiving end short-circuited. Obviously, branch C, being short-circuited, its effect is eliminated, and the equivalent Π circuit is

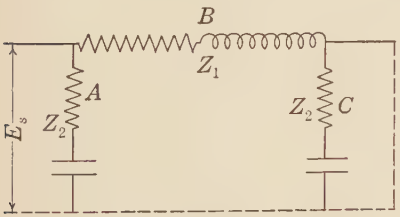

Fig. 56

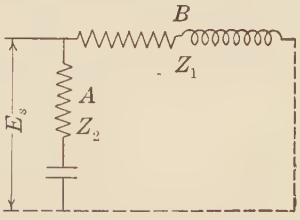

Fig. 57

reduced to that given in Fig. 57. By inspection, the sending-end impedance is

$$Z_\pi = Z_s = \frac{1}{\frac{1}{Z_1} + \frac{1}{Z_2}} = \frac{Z_1 Z_2}{Z_1 + Z_2}. \tag{62}$$

Substituting for Z_1 and Z_2 their values in terms of the characteristic line impedance and propagation factor, we get

$$Z_s = Z_0 \tanh pS,$$

which is the same as (3), Chapter VII.

The sending-end current is

$$I_s = \frac{E_s}{Z_s} = E_s \frac{Z_1 + Z_2}{Z_1 Z_2} = \frac{E_s}{Z_0 \tanh pS}, \tag{63}$$

which is the same as (4), Chapter VII.

The current at the short-circuited end is

$$I_r = I_B = \frac{E_s}{Z_1} = \frac{E_s}{Z_0 \sinh pS}, \tag{64}$$

which is the same as (6), Chapter VII.

96. The Equivalent Π Circuit of a Line with any End Impedance $Z_r e^{j\theta_r}$. — The diagram of the equivalent Π circuit of such a line is shown in Fig. 58. By inspection, it will be seen that the sending-end

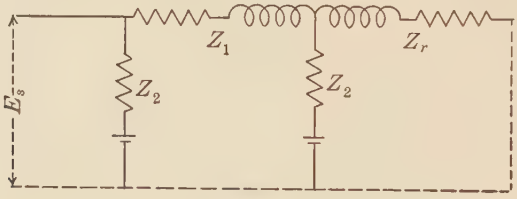

Fig. 58

impedance of the line which this equivalent Π circuit represents is

$$Z_\pi = Z_s = \frac{1}{\dfrac{1}{Z_2} + \dfrac{Z_r + Z_2}{Z_r Z_1 + Z_1 Z_2 + Z_2 Z_r}},$$

or

$$Z_\pi = \frac{Z_1 Z_2^2 + Z_r (Z_1 Z_2 + Z_2^2)}{Z_r (Z_1 + 2 Z_2) + Z_2 (Z_1 + Z_2)}. \tag{65}$$

This is equivalent to (5), Chapter VIII, as may be proved by substituting for the branch impedances their values in terms of the characteristic impedance and the propagation factor, and by properly transforming it.

CHAPTER XII

THE PHENOMENA OF DISTORTION — DISTORTIONLESS TRANSMISSION

It is evident that the discussion of electric-wave transmission given in the preceding eight chapters is applicable to heavy electrical transmission of power as well as to telephonic transmission. The development of the theory was based on the hypothesis that the voltage and the resulting currents are harmonic functions of time, following the pure sine law, and the formulas obtained are true for one frequency only.

Telephonic waves, *i.e.*, waves generated by the telephone transmitter actuated by the human voice, are, however, complex waves. A multitude of different frequencies, ranging from about 200 to about 2400 cycles per second, are superimposed on each other and transmitted simultaneously. Evidently, the problem of transmission as applied to telephony would be a very complicated one, if the complex waves of voltages and currents were considered. In practical calculations, as pointed out elsewhere, the problem is greatly simplified by assuming that a voice or telephonic wave is a pure sine wave effectively equivalent to the complex voice wave, and of a frequency equal to the mean frequency encountered in telephony, the accepted value of which is 796 cycles per second, corresponding to a mean angular velocity of $\omega = 5000$ radians per second. This value of ω is used in many telephonic calculations and the results are sufficiently accurate for most practical purposes, provided that line corrective apparatus is used to maintain the form of the wave, however complex it may be, during its travel along the line.

97. Variation of the Characteristic Line Impedance with Frequency. Aerial Lines. — A line with definite physical constants has a characteristic line impedance Z_0 given by the formula

$$Z_0 = \sqrt[4]{\frac{R^2 + L^2\omega^2}{G^2 + C^2\omega^2}}\, \epsilon^{\frac{j(\alpha_1 - \alpha_2)}{2}}, \tag{1}$$

where

$$\alpha_1 = \tan^{-1}\frac{L\omega}{R},$$

and

$$\alpha_2 = \tan^{-1}\frac{C\omega}{G}.$$

From (1) we see that the modulus of the impedance as well as its argument depend upon the frequency. When the frequency approaches zero, both α_1 and α_2, and hence the argument of Z_0, approach zero, while the modulus of Z_0 approaches the constant value

$$Z_0 = \sqrt{\frac{R}{G}}, \tag{2}$$

which is the characteristic resistance of a direct-current line.

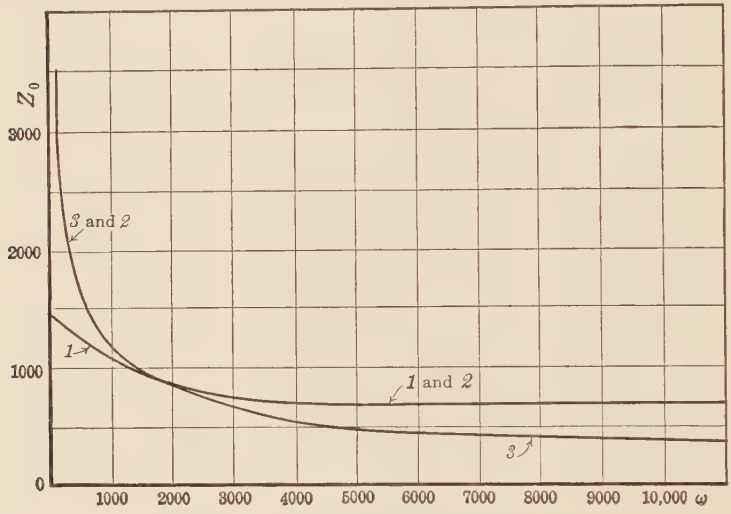

FIG. 59. — Variation of Z_0 with ω

When the frequency is very high, tending toward infinity, both α_1 and α_2 approach 90°; and the characteristic phase angle α approaches zero. If we write equation (1) in the form

$$Z_0 = \sqrt[4]{\frac{\frac{R^2}{\omega^2} + L^2}{\frac{G^2}{\omega^2} + C^2}},$$

we see that the modulus of the characteristic impedance approaches the constant value

$$Z_0 = \sqrt{\frac{L}{C}} \tag{3}$$

which is the natural impedance, or the surge impedance, of the line. Curve 1, Fig. 59, shows the variation of Z_0 with ω, and curve 1,

Fig. 60, shows the variation of the phase angle α with ω for an aerial line of definite physical constants. The dissipative component of the characteristic impedance $Z_0 \cos \alpha$, as well as the reactive component $Z_0 \sin \alpha$, will also vary with the frequency, due to the dependence of both Z_0 and α on the frequency.

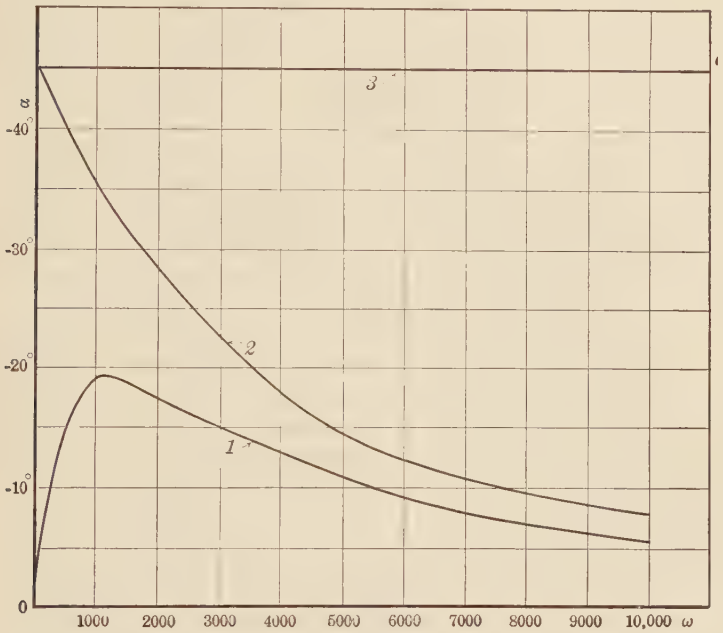

Fig. 60. — Variation of Characteristic Phase Angle with ω

It follows from what precedes that for each component harmonic of the complex telephonic wave there will be a corresponding characteristic impedance, whose value for a line of definite electrical constants is entirely determined by the frequency of the harmonic.

98. Variation of the Characteristic Impedance with Frequency when Leakage Conductance is neglected. — In the case of a lead-sheathed cable, the leakage conductance G may be neglected, provided that the dielectric hysteresis is small. The characteristic impedance for a line of this type, when $G = 0$, is

$$Z_0 = \sqrt[4]{\frac{R^2 + L^2\omega^2}{C^2\omega^2}}\, e^{j\alpha}, \tag{4}$$

where

$$\alpha = \frac{\alpha_1 - 90^\circ}{2},$$

since

$$\alpha_2 = \tan^{-1}\frac{C\omega}{0} = 90^\circ$$

If we write equation (4) in the form

$$Z_0 = \sqrt[4]{\frac{\frac{R^2}{\omega^2} + L^2}{C^2}},$$

we see that Z_0 varies between infinity and $Z_0 = \sqrt{L/C}$ as ω is made to vary between zero and infinity. Curve 2, Fig. 59, shows the law of this variation. The characteristic phase angle varies between the values $\alpha = -45^\circ$ and $\alpha = 0^\circ$, as ω changes from $\omega = 0$ to $\omega = \infty$. Curve 2, Fig. 60, shows the variation of α with ω.

It may be concluded that for lines for which the leakage conductance is negligibly small, the characteristic impedance pertaining to the lower harmonics will be correspondingly higher than for lines with appreciable leakage conductance and with the same resistance, inductance, and capacitance per mile of line. The leakage conductance G, for the higher harmonics, may be neglected without serious error.

99. Variation of Characteristic Impedance with Frequency when G and L are both Neglected. — The inductance may also be negligibly small, especially in cable lines. Neglecting its value, and that of the leakage, in the calculation of the characteristic impedance, we find

$$Z_0 = \sqrt{\frac{R}{C\omega}}\epsilon^{-j45^\circ}, \tag{5}$$

which shows that Z_0 varies inversely with the square root of the frequency, while the characteristic phase angle remains constant for all values of the frequency. This may be seen from the fact that

$$\alpha_1 = \tan^{-1}\frac{L\omega}{R} = \tan^{-1} 0 = 0,$$

and

$$\alpha_2 = \tan^{-1}\frac{C\omega}{0} = 90^\circ.$$

It follows from what precedes that, for lines of this type, the variation of Z_0 for the different harmonics of the complex telephonic wave is much more prominent, as compared with the two cases previously discussed. This is illustrated by curve 3, Fig. 59.

100. Causes of Wave Distortion. — The propagation constant of any line is given by the formula

$$p = \sqrt[4]{(R^2 + L^2\omega^2)\,(G^2 + C^2\omega^2)}\;\epsilon^{j\delta}, \tag{6}$$

where

$$\delta = \frac{\tan^{-1}\dfrac{L\omega}{R} + \tan^{-1}\dfrac{C\omega}{G}}{2}.$$

From (6), we see that both the modulus of p and its argument vary with the frequency. Since the component harmonics of the complex telephonic waves are of different frequencies, it is evident that there will be a different propagation constant for each harmonic. Moreover, the real component of the propagation constant, the attenuation constant a, and the imaginary component, the wave-length constant v, will also depend upon the frequency. Hence each harmonic component of the complex telephonic wave will attenuate at a different rate, and will travel at a different velocity.

This unequal rate of attenuation and velocity of propagation of the components of the voice wave results in a gradual and increasing change in the original shape of the composite wave as it travels along the line. This change in the shape of the wave is called ***distortion.*** It is evident that such a distorted wave, on reaching the receiver, will not reproduce the sounds that generated the original electric wave. The quality of the reproduced sounds is therefore altered, resulting in indistinctness of the reproduced speech. This distortion of the wave, becoming more and more pronounced as the line-length is increased, would naturally limit the distance over which telephonic reproduction of speech could be transmitted with distinctness, if means were not employed for maintaining the wave-shape more or less intact. A line with corrective apparatus for this purpose is said to be a ***distortionless line.***

101. Variation of the Propagation Constant with Frequency. Aerial Lines. — Equation (6) shows that the propagation constant has a minimum value

$$p = \sqrt{RG}\;\epsilon^{j0^\circ} \tag{7}$$

when $\omega = 0$. It is evident that the attenuation constant, for this value of frequency, is

$$a = p \cos \delta = p. \tag{8}$$

and that the wave-length constant is

$$v = p \sin \delta = 0. \tag{9}$$

While the modulus of p increases with ω, as shown in curve marked p, Fig. 61, its argument tends toward the constant value $\delta = 90°$, as shown by curve marked δ, Fig. 61. Since cos δ decreases and sin δ increases with ω, both the attenuation constant

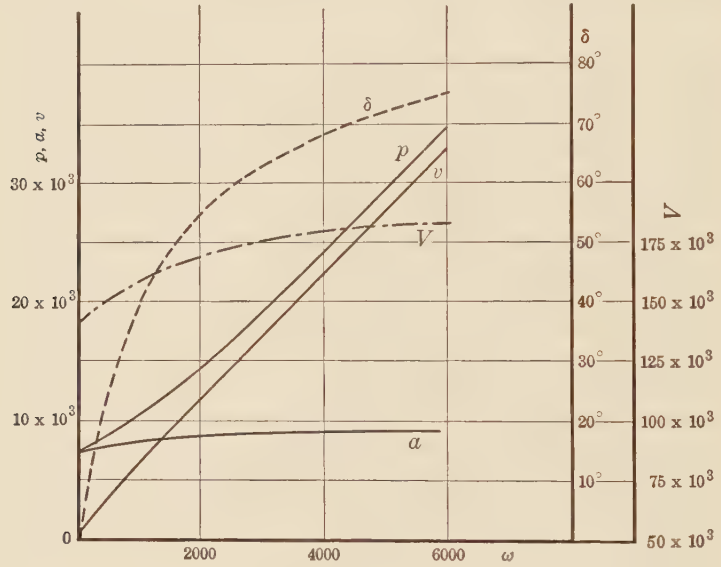

Fig. 61. — Variation of p, a, v, δ, and V with ω. Aerial Line

and wave-length constant will increase with the frequency, but not at the same rate. The wave-length constant increases much faster than the attenuation constant, as is shown by the curves marked a and v, Fig. 61. The scales for p, a and v in this figure are respectively in terms of complex hyperbolic radians, real hyperbolic radians and real circular radians, while the scale for V is in terms of miles per second.

It will be noted that, for lines of this type, the attenuation constant does not vary appreciably with frequencies higher than the accepted average of 796 cycles per second. Hence, the high-

frequency components of the complex telephonic wave attenuate at about the same rate, while the rate of attenuation of the component harmonics of frequencies less than the average is lower the lower the frequency. Differences in the velocity of propagation are also more marked for the lower harmonics, for with increase in frequency the velocity of propagation tends toward the constant velocity of light.

102. Variation of the Propagation Constant with Frequency for Lines with Negligible Leakage Conductance. — If we set $G = 0$ in (6), the value of p becomes

$$p = \sqrt{(R + jL\omega)\, jC\omega},$$

or

$$p = \sqrt{C\omega\, \sqrt{R^2 + L^2\omega^2}\epsilon^{j\delta}}, \tag{10}$$

where

$$\delta = \frac{\tan^{-1} \frac{L\omega}{R} + 90^\circ}{2}.$$

It is evident that the limiting values of p are zero and infinity for values of ω equal to zero and infinity, respectively, as shown by curve marked p, Fig. 62. The argument δ changes from 45° to 90° as ω varies from zero to infinity, as indicated by curve marked δ, Fig. 62.

For any value of ω the attenuation constant is

$$a = \sqrt{C\omega \sqrt{R^2 + L^2\omega^2}} \cos \frac{\alpha_1 + 90^\circ}{2}.$$

But we have

$$\cos \frac{\alpha_1 + 90^\circ}{2} = \sqrt{\tfrac{1}{2}\{1 + \cos(\alpha_1 + 90^\circ)\}} = \sqrt{\tfrac{1}{2}(1 - \sin \alpha_1)},$$

and

$$\sin \alpha_1 = \frac{L\omega}{\sqrt{R^2 + L^2\omega^2}}$$

so that

$$\cos \frac{\alpha_1 + 90^\circ}{2} = \sqrt{\frac{1}{2}\left(1 - \frac{L\omega}{\sqrt{R^2 + L^2\omega^2}}\right)}.$$

Substituting in the expression for a, we get

$$a = \sqrt{C\omega \sqrt{R^2 + L^2\omega^2}} \times \frac{\sqrt{R^2 + L^2\omega^2} - L\omega}{2\sqrt{R^2 + L^2\omega^2}},$$

which reduces to

$$a = \sqrt{\frac{C\omega}{2}\{\sqrt{R^2 + L^2\omega^2} - L\omega\}}. \tag{11}$$

The curve marked a, Fig. 62, shows the variation of the attenuation constant a with the frequency. As in the cases previously discussed, its variation is more marked for frequencies lower than the average telephonic frequency. Hence the lower harmonics of the telephonic wave will differ greatly in attenuation as compared with the higher harmonics, and more so than for the aerial line.

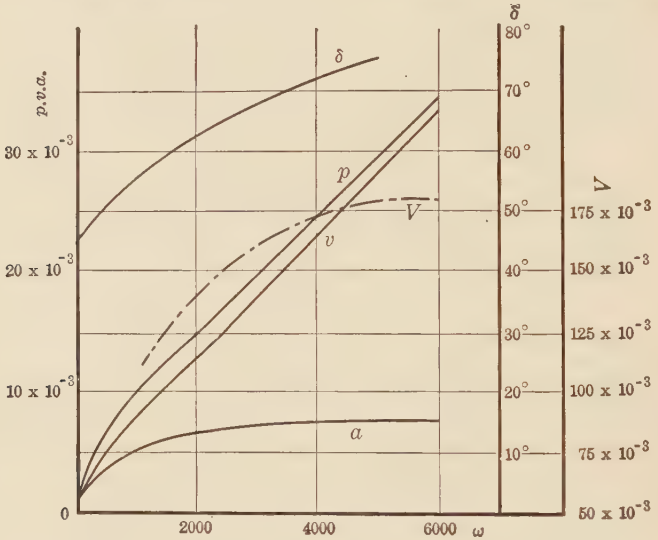

FIG. 62. — Variation of p, a v, δ and V with ω, for $G = 0$

The wave-length constant v may be obtained in a manner similar to that used in the determination of the attenuation constant. This gives

$$v = \sqrt{\frac{C\omega}{2}\left\{\sqrt{R^2 + L^2\omega^2} + L\omega\right\}} \qquad (12)$$

The curve marked v, Fig. 62, shows the variation of the wave-length constant with the frequency, and curve marked V the variation of the velocity of propagation with the frequency. The curve V indicates that the velocity of the low-frequency components of the complex telephonic wave is decidedly lower than that of the high-frequency harmonics.

103. Variation of the Propagation Constant with Frequency for Lines with Negligible Leakage and Inductance. — Both leakage conductance and the inductance are negligibly small for cable

lines. If we set both L and G equal to zero, the expression for the propagation constant reduces to

$$p = \sqrt{C\omega R}\, \epsilon^{j45°}. \tag{13}$$

The argument of the propagation constant is constant at 45°, no matter what the frequency may be, as seen from the following relations:

$$\alpha_1 = \tan^{-1} \frac{L\omega}{R} = 0, \quad \text{when } L = 0,$$

and

$$\alpha_2 = \tan^{-1} \frac{C\omega}{G} = 90°, \text{ when } G = 0.$$

Under the assumption that both inductance and leakage are negligibly small, the propagation constant varies directly as the square root of the frequency. Since the argument is constant at 45°, the attenuation constant and the wave-length constants are equal to one another for any value of ω:

$$a = v = p \cos 45° = p \sin 45° = \sqrt{\frac{C\omega R}{2}}. \tag{14}$$

While expression (5) for the characteristic impedance, and expression (13) for the propagation constant of cables with negligible inductance and leakage conductance are frequently used in practical calculations, it must be kept in mind that they are only approximately correct within definite limits of frequency. The inductance, no matter how small, must be taken into account for the high-frequency harmonics; otherwise erroneous results might be obtained. Thus expression (14) shows that with $L = 0$, the wave-length constant, being directly proportional to the square root of the frequency, may give calculated velocities of propagation for high-frequency harmonics higher than that of light, which is impossible. That the inductance, no matter how small it may be in comparison with the other line constants, must be taken into account, follows also from the fact that a system with zero inductance is incapable of transferring electric energy.

104. Conditions for Distortionless Transmission of Telephonic Waves. — If the harmonic components of a complex telephonic wave should all encounter the same impedance, attenuate at the same rate, and travel with the same velocity, the shape of the

complex wave would evidently be maintained during its propagation. Hence the problem of distortionless transmission is to find the conditions to be imposed upon a line, which would make the characteristic impedance, the attenuation, and the velocity of propagation independent of the frequency.

Consider the general expression for the characteristic impedance of any line

$$Z_0 = \sqrt{\frac{R + jL\omega}{G + jC\omega}}.$$

In order that Z_0 be independent of ω, the derivative of Z_0 with respect to ω must be identically equal to zero, $dZ_0/d\omega \equiv 0$. In other words, curve 1, Fig. 59, must be a straight line parallel to the axis of ω. Differentiating the preceding expression for Z_0 with respect to ω, and equating the result to zero, we get

$$RC = LG. \tag{15}$$

This relation between the line constants must therefore be satisfied if the line is to transmit complex telephonic waves without distortion. Obviously the last expression may also be written in the form

$$\frac{R}{G} = \frac{L}{C}. \tag{16}$$

Substituting for any one of the line constants in terms of the other three, the expression for the characteristic impedance for distortionless transmission may be written in the form

$$Z_0 = \sqrt{\frac{R}{G}} = \sqrt{\frac{L}{C}}. \tag{17}$$

This last expression shows that the characteristic impedance of a distortionless line has the properties of a direct-current line, the energy transfer being with no reactive component, *i.e.*, the transmission being of unity power-factor. This is readily seen from the fact that

$$\tan \alpha_1 = \frac{L\omega}{R} = \tan \alpha_2 = \frac{C\omega}{G},$$

whence $\alpha = 0$ and $\cos \alpha = 1$.

The propagation constant of a line on which the distortionless condition (15) is imposed may be obtained by substituting for any of the four line constants in terms of the other three in the

general expression for it. Thus, substituting $C = LG/R$ in the expression for p, we get

$$p = \sqrt[4]{(R^2 + L^2\omega^2)\left(G^2 + \frac{L^2G^2\omega^2}{R^2}\right)}\,\epsilon^{j\alpha_1},$$

$$= \sqrt[4]{\left(RG + \frac{G}{R}L^2\omega^2\right)^2}\,\epsilon^{j\alpha_1} = \sqrt{\frac{G}{R}}\sqrt{(R^2 + L^2\omega^2)}\,\epsilon^{j\alpha_1}.$$

Using the trigonometric form, we find

$$p = \sqrt{\frac{G}{R}(R^2 + L^2\omega^2)}\,(\cos\alpha_1 + j\sin\alpha_1).$$

Since

$$\cos\alpha_1 = \frac{R}{\sqrt{R^2 + L^2\omega^2}}, \text{ and } \sin\alpha_1 = \frac{L\omega}{\sqrt{R^2 + L^2\omega^2}},$$

the expression for p becomes

$$p = \sqrt{\frac{G}{R}(R^2 + L^2\omega^2)}\left\{\frac{R}{\sqrt{R + jL^2\omega^2}} + j\frac{L\omega}{\sqrt{R^2 + L^2\omega^2}}\right\}$$

which simplifies to

$$p = \sqrt{RG} + jL\omega\sqrt{\frac{G}{R}},$$

or, by equation (16),

$$p = \sqrt{RG} + j\omega\sqrt{LC}. \tag{18}$$

Since $p = a + jv$, it follows that the attenuation constant a and the wave-length constant v, respectively, are

$$a = \sqrt{RG}, \tag{19}$$

and

$$v = \omega\sqrt{LC}, \tag{20}$$

and the velocity of propagation is

$$V = \frac{\omega}{v} = \frac{1}{\sqrt{LC}}, \tag{21}$$

which is independent of the frequency.

Hence waves of any frequency whatever, when transmitted over a line whose electrical constants bear to one another the preceding relation $RC = LG$, will encounter the same impedance, attenuate at the same rate, and travel with the same velocity. It follows that a complex wave will maintain its shape during its propagation, and hence the line is a *distortionless line.*

105. Methods of varying the Line Constants. Variation of RC. — Since the quantity RC is larger than LG in the ordinary telephone line, distortionless transmission might be obtained by a reduction of RC, either by decreasing the capacitance or the resistance per mile of line. The capacitance of a two-wire aerial line per mile of line is given by the expression

$$C = \frac{.01941\, k}{\log_{10} \frac{2D}{d}} \text{ microfarads,} \tag{22}$$

where

D = interaxial distance between the conductors,
d = diameter of the conductors, measured in the same units as D,
k = the specific inductive capacity, which in air is equal to unity.

The above expression indicates that a reduction in the capacitance may be effected either by increasing the interaxial distance D between the conductors, or by decreasing the size of the conductors. It is clear, however, that on account of the rather large number of wires carried on the same pole, it would be impractical to increase the distance between the conductors. Decreasing the capacitance by a reduction in the size of conductor d will be associated with a corresponding increase in resistance. This would not only defeat the purpose in view, but would actually produce an increase in the attenuation and in the amount of energy loss in the line, and thereby reduce the efficiency of transmission.

The conclusions thus derived for the two-wire aerial line hold also for a single overhead conductor with ground return, since the capacitance per mile of such a line is

$$C = \frac{.03882}{\log_{10} \frac{h}{d}} \text{ microfarads,} \tag{23}$$

where

h = height of conductor above the ground, and
d = diameter of conductor, measured in the same units.

It is evident that the capacitance of this type of line could be decreased by increasing the distance of the conductor from the ground. This would be impracticable, since the height at which the wire may be strung above the ground is rather limited by length of pole and also by the number of conductors carried by the same pole.

In the twin conductor underground cable with grounded sheath, the capacitance is rather high. The wires are separated by a dielectric other than air, and they are naturally not only close together, but are of a minimum size commensurate with good economy.

The capacitance of such a cable per mile of line is approximately

$$C = \frac{.01941\, k}{\log \frac{2\, D\, (\mathcal{D}^2 - D^2)}{d\, (\mathcal{D}^2 + D^2)}}, \text{ microfarads} \tag{24}$$

where

D = interaxial distance between the conductors,
$\mathcal{D}$ = inner diameter of the lead sheath, and
d = the diameter of the conductor, all measured in the same units.

It is evident that the quantities D, $\mathcal{D}$, and d are limited in their values. Hence the only practicable possibility of decreasing the capacitance is by decreasing the specific inductive capacity k, through the use of loosely wrapped dry paper spirally wound on the wires. The reduction in capacitance by this method, however, is not at all sufficient to bring about the conditions necessary for distortionless transmission, for the values of the inductance and the leakage conductance of a cable line are very small.

The quantity RC might also be reduced by decreasing the resistance per mile of line. This may easily be accomplished by the use of a larger size conductor. As shown by equation (22), this will produce the undesirable effect of also increasing the capacitance. For long aerial lines, it is advantageous to use the maximum size wire consistent with good economy, the object being rather a reduction of the energy loss in the line than for the purpose discussed here.

106. Variation of LG. — The conditions to be imposed upon a line for distortionless transmission of complex telephonic waves may be obtained also by increasing the quantity LG, either by increasing the leakage conductance or the inductance per mile of line.

Leakage conductance, which is the reciprocal of the insulation resistance, depends in aerial lines on the type of insulator, pins and cross-arms, as well as on the degree of atmospheric humidity. Thus the leakage conductance of an aerial line may vary between .001 to .002 micromhos, corresponding to 1000 to 500 megohms per mile, in dry weather. In average weather the leakage conductance may vary between .04 to .08 micromhos, corresponding

to 25 to 12.5 megohms per mile, while in very wet weather it may be as high as .5 to 1.0 micromhos, corresponding to 2 to 1 megohms per mile of line.

In underground cables covered with a lead sheath, with the conductors properly insulated from each other, and not subject to weather conditions, the leakage conductance is low, but its effective value is greatly increased by dielectric hysteresis. This dielectric hysteresis varies almost directly with the frequency, and may be equivalent to as much as 2 micromhos leakage conductance per mile of cable. While an increase in leakage conductance will improve transmission, inasmuch as it will lessen distortion, it will do so at the expense of efficiency, since the attenuation and consequently energy dissipation in the line is increased.

The most practical and economic method of bringing about the condition for distortionless transmission is to increase the inductance L. The inductance per mile of line is

$$(25) \qquad L = 2\left(.08047\,\mu + .74113 \log_{10} \frac{2D}{d}\right) \text{ milli-henries,}$$

where

D = interaxial distance between the conductors,
d = diameter of the conductors, measured in the same units, and
μ = the permeability of the conductor material.

It will be observed, by an inspection of this expression, that the inductance may be increased by increasing the distance between the conductors. This would be doubly beneficial, since the capacitance would also be diminished thereby. However, as was mentioned in discussing methods of changing the capacitance, it is not practical to depart from the standard spacing of about 12 inches for aerial lines, on account of the large number of circuits carried on the same pole and also on account of the bad inductive disturbances which may result from wider spacing.

The inductance may also be increased by decreasing the size of the conductors, but this is not economical since the resistance is thereby increased and the efficiency of transmission reduced. The inductance may also be increased by the use of a conductor of permeability larger than unity, *i.e.*, by the use of iron conductors, but the size of the wire will have to be correspondingly increased if the efficiency of transmission is not to be greatly affected. The use of iron conductors will increase the effective ohmic resistance,

on account of the higher resistivity of iron as compared with that of copper, on account of the larger skin effect, and also on account of the equivalent resistance of the eddy current loss and the hysteresis loss in the conductor.

The most effective method of bringing about the distortionless condition of transmission is to add inductance to the line. If the added inductance is uniformly distributed, the line is said to be *uniformly loaded*, as in the *Krarup system* for ocean telephony. If the added inductance is lumped at regular intervals along the line, the loading is said to be *regularly lumped* or *concentrated loading*, as in the *Pupin system of* loading telephone lines.

CHAPTER XIII

THE INDUCTIVELY LOADED LINE

107. Pupin System of Loading. — If the necessary inductance to bring about distortionless transmission is uniformly distributed and does not in any way change the values of the other line constants, the inductive loading of the line is said to be ***ideal,*** because this condition can never be fully realized in practice. Pure inductance cannot be added to any line without increasing the effective value of the resistance on account of the actual increase in ohmic resistance and on account of the equivalent resistance of the added losses due to hysteresis and eddy currents induced in the magnetic circuit of the added inductance. If the inductance is in the form of low resistance impedance coils that are inserted in the line at regular and definite maximum distances from one another, we shall see that the line will behave very nearly like a uniformly loaded one. This scheme was suggested by M. I. Pupin of Columbia University, and is called the *Pupin system of loading.*

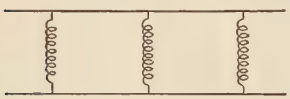

Fig. 63

If the spacing distance between the added inductive impedances, generally called *loading coils,* is the same all along the line, the line is said to be a *regularly loaded line.* If the loading coils are connected in parallel across the line conductors, the

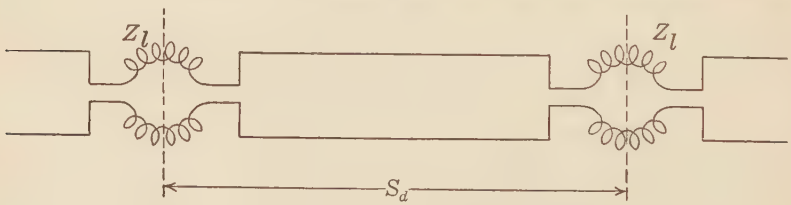

Fig. 64

loading is said to be ***leak loading*** or ***parallel loading,*** shown diagrammatically in Fig. 63. If the loading coils are connected in series with the line conductors as shown in Fig. 64, the loading is called ***series***

loading, and the line is said to be a *regularly series-loaded line.* Regular series loading is universally used to eliminate distortion. Irrespective of the method of loading, the lumping of low resistance impedance coils at regular intervals along the line will naturally disturb the uniform distribution of the line constants, with the result that the line will behave somewhat differently from the smooth or uniform line.

The behavior of a line so loaded may be investigated, however, through its equivalent circuit, the equivalent *T* circuit being best adapted to the study of the series-loaded line and the equivalent Π circuit to the leak-loaded or parallel-loaded line.

108. The Equivalent Circuit of the Regularly Series-Loaded Line. — Regular series loading consists in inserting low resistance impedance coils of definite amount of self-induction, at regular intervals, in each of the line conductors. Consider a line section of length S_d of a non-loaded line of length S miles, with uniformly distributed electrical constants R_0, L_0, C_0, and G_0 per mile of line. The characteristic impedance and the propagation constant of this line will be, respectively,

$$(1) \qquad Z_0 = \sqrt{\frac{R_0 + jL_0\omega}{G_0 + jC_0\omega}},$$

$$(2) \qquad p_0 = \sqrt{(R_0 + jL_0\omega)\,(G_0 + jC_0\omega)}.$$

The equivalent *T* circuit of a section of length S_d of this line is shown in Fig. 65, in which the branch impedances, by equations (21) and (24), Chapter XI, are

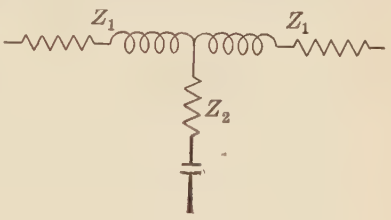

Fig. 65

$$(3) \qquad Z_1 = Z_0 \tanh \frac{p_0 S_d}{2},$$

$$(4) \qquad Z_2 = \frac{Z_0}{\sinh p_0 S_d}$$

Let induction coils of equal impedance Z_l be inserted at the ends of the conductors of the line-section as shown diagrammatically in Fig. 64. It might be assumed that only one-half of each of the four impedance coils is affecting the line section under consideration. However, one-half of each of the coils to the right is affecting the section on the right, and one-half of the coils on the left is affecting the section to the left of the line-section under consideration.

The lump impedance per section is therefore 2 Z_l, one-half of which is at each end of the section. The equivalent T circuit of the section so loaded is shown in Fig. 66.

It will be noted that the loading of the line-section with impedance coils Z_l at its ends does not alter the value of the leak impedance

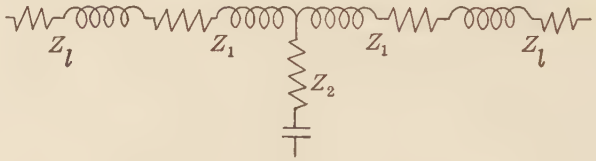

FIG. 66

Z_2 of its equivalent T circuit. Consequently, each section of length S_d of the line may be represented by the same equivalent circuit.

109. The Uniform Line Equivalent to the Regularly Series-Loaded Line. — Having obtained the equivalent T circuit of the loaded section, it is possible to find a uniform line of length equal to the section and equivalent to its T circuit, *i.e.*, equivalent to the loaded section itself. This enables us to investigate the behavior of the loaded line in terms of a uniform or smooth line with distributed electrical properties, differing from, but expressed in terms of the transmission constants of original line before loading, and in terms of the loading-coil impedance.

The process of investigation will consist then in: (1) converting the section of the uniform non-loaded line into its T equivalent circuit; (2) converting this equivalent T circuit into a new T circuit representing the loaded section; and (3) finding a uniform line equivalent to the T circuit of the loaded section, *i.e.*, equivalent to the loaded line-section itself.

If Z_{q0} and p_q are the characteristic impedance and the propagation constant, respectively, of the uniform line effectively equivalent to the T circuit of the loaded line, we have, by equations (21) and (24), Chapter XI,

$$Z_l + Z_1 = Z_{q0} \tanh \frac{p_q S_d}{2} \tag{5}$$

and

$$Z_2 = \frac{Z_{q0}}{\sinh p_q S_d} \tag{6}$$

where the subscript l refers to the loading coils, and q to the smooth

line, equivalent to the loaded line. Substituting in equation (5) the value of Z_1 from (3), we get

$$Z_l + Z_0 \tanh \frac{p_0 S_d}{2} = Z_{q0} \tanh \frac{p_q S_d}{2},$$

whence

$$\tanh \frac{p_q S_d}{2} = \frac{Z_l}{Z_{q0}} + \frac{Z_0}{Z_{q0}} \tanh \frac{p_0 S_d}{2}. \tag{7}$$

Substituting in equation (6) the value of Z_2 from (4), we obtain

$$\frac{Z_0}{\sinh p_0 S_d} = \frac{Z_{q0}}{\sinh p_q S_d}, \tag{8}$$

whence

$$Z_{q0} = Z_0 \frac{\sinh p_q S_d}{\sinh p_0 S_d}. \tag{9}$$

Substituting this value of Z_{q0} in equation (7), we get

$$\tanh \frac{p_q S_d}{2} = \left[Z_l + Z_0 \tanh \frac{p_0 S_d}{2} \right] \frac{\sinh p_0 S_d}{Z_0 \sinh p_q S_d}, \tag{10}$$

or

$$\sinh p_q S_d \tanh \frac{p_q S_d}{2} = \sinh p_0 S_d \left[\frac{Z_l}{Z_0} + \tanh \frac{p_0 S_d}{2} \right]. \tag{11}$$

But we have

$$\tanh \frac{p_q S_d}{2} = \frac{\sinh p_q S_d}{1 + \cosh p_q S_d},$$

and

$$\tanh \frac{p_0 S_d}{2} = \frac{\sinh p_0 S_d}{1 + \cosh p_0 S_d}.$$

Hence, substituting in equation (11), and multiplying out, we obtain

$$\frac{\sinh^2 p_q S_d}{1 + \cosh p_q S_d} = \frac{Z_l}{Z_0} \sinh p_0 S_d + \frac{\sinh^2 p_0 S_d}{1 + \cosh p_0 S_d}, \tag{12}$$

or

$$\frac{\cosh^2 p_q S_d - 1}{\cosh p_q S_d + 1} = \frac{Z_l}{Z_0} \sinh p_0 S_d + \frac{\cosh^2 p_0 S_d - 1}{\cosh p_0 S_d + 1}. \tag{13}$$

Since

$$\cosh^2 p_q S_d - 1 = (\cosh p_q S_d + 1)(\cosh p_q S_d - 1),$$

it follows that expression (13) may be written

$$\cosh p_q S_d - 1 = \frac{Z_l}{Z_0} \sinh p_0 S_d + \cosh p_0 S_d - 1, \tag{14}$$

whence

$$\cosh p_q S_d = \frac{Z_l}{Z_0} \sinh p_0 S_d + \cosh p_0 S_d. \tag{15}$$

This last equation gives the hyperbolic cosine of the propagation factor of the uniform line equivalent to the *T* circuit of the regularly series-loaded line, *i.e.*, of the loaded line itself, in terms of the constants of the uniform line before loading and of the loading-coil impedance.

The hyperbolic sine of the propagation factor of the uniform line equivalent to the loaded line may be obtained by substituting $\sinh^2 p_q S_d + 1$ for $\cosh^2 p_q S_d$ in equation (15). This gives

$$1 + \sinh^2 p_q S_d = \frac{Z_l^2}{Z_0^2} \sinh^2 p_0 S_d + \cosh^2 p_0 S_d + \frac{2 Z_l}{Z_0} \sinh p_0 S_d \cosh p_0 S_d,$$

or

$$\sinh^2 p_q S_d = \frac{Z_l^2}{Z_0^2} \sinh^2 p_0 S_d + \sinh^2 p_0 S_d + \frac{Z_l}{Z_0} \sinh 2 p_0 S_d,$$

or

$$\sinh p_q S_d = \sqrt{\sinh^2 p_0 S_d \left(\frac{Z_l^2}{Z_0^2} + 1\right) + \frac{Z_l}{Z_0} \sinh 2 p_0 S_d}, \tag{16}$$

which may also be written in the form

$$\sinh p_q S_d = \sinh p_0 S_d \sqrt{1 + \frac{Z_l^2}{Z_0^2} + \frac{2 Z_l}{Z_0} \operatorname{ctnh} p_0 S_d}. \tag{17}$$

110. Characteristic Impedance of the Regularly Series-Loaded Line. — Substituting in equation (9) the value of the hyperbolic sine of the propagation factor of the loaded line from (17), we get

$$Z_{q0} = Z_0 \sqrt{1 + \frac{Z_l^2}{Z_0^2} + \frac{2 Z_l}{Z_0} \operatorname{ctnh} p_0 S_d},$$

or

$$Z_{q0} = \sqrt{Z_0^2 + Z_l^2 + 2 Z_0 Z_l \operatorname{ctnh} p_0 S_d}, \tag{18}$$

which gives the characteristic line impedance of the uniform line equivalent to the regularly series-loaded line, in terms of quantities pertaining to the uniform line before loading and of the loading-coil impedance.

111. Determination of the Propagation, Attenuation, and Wave-Length Constants of the Regularly Series-Loaded Line. — The values of p_q, a_q, and v_q of the regularly series-loaded line may be obtained from equation (15) in the following manner. Since the numerical values of the hyperbolic sine and cosine of the propaga-

tion factor of the uniform line before loading are supposed to be known, they may be written in the form

$$\sinh p_0 S_d = A_1 \epsilon^{j\beta_2}, \tag{19}$$

$$\cosh p_0 S_d = B \epsilon^{j\beta_1}. \tag{20}$$

Assigning arguments to the loading-coil impedance and to the characteristic impedance of the line before loading, equation (15) may be written in the form

$$\cosh (a_q + jv_q) S_d = \frac{Z_l \epsilon^{j\theta_l}}{Z_0 \epsilon^{j\alpha}} A_1 \epsilon^{j\beta_2} + B \epsilon^{j\beta_1}, \tag{21}$$

where θ_l is the phase angle of the loading-coil impedance. If we set for brevity

$$\frac{Z_l}{Z_0} A_1 = A,$$

the preceding expression becomes

$$\cosh (a_q + jv_q) S_d = A \epsilon^{j(\theta_l - \alpha + \beta_2)} + B \epsilon^{j\beta_1}. \tag{22}$$

Expanding the left side, and using the trigonometric form for the terms on the right, we obtain

$$\cosh a_q S_d \cos v_q S_d + j \sinh a_q S_d \sin v_q S_d = A \cos (\theta_l - \alpha + \beta_2) + B \cos \beta_1 + j [A \sin (\theta_l - \alpha + \beta_2) + B \sin \beta_1]. \tag{23}$$

Equating the reals and imaginaries, respectively, on both sides of the equality sign, and denoting the real term on the right by M and the imaginary term by N, we find

$$\cosh a_q S_d \cos v_q S_d = M \tag{24}$$

$$\sinh a_q S_d \sin v_q S_d = N. \tag{25}$$

Squaring on both sides, and substituting

$$\sinh^2 a_q S_d = \cosh^2 a_q S_s - 1, \tag{26}$$

and

$$\sin^2 v_q S_d = 1 - \cos^2 v_q S_d, \tag{27}$$

We may write equations (24) and (25), respectively, in the forms

$$\cosh^2 a_q S_d \cos^2 v_q S_d = M^2, \tag{28}$$

$$(\cosh^2 a_q S_d - 1)(1 - \cos^2 v_q S_d) = N^2. \tag{29}$$

If we set, for brevity,

$$\cosh^2 a_q S_d = x, \tag{30}$$

and

$$\cos^2 v_q S_d = y, \tag{31}$$

equations (28) and (29) become

$$xy = M^2, \tag{32}$$

and

$$(x - 1)(1 - y) = N^2. \tag{33}$$

These equations may now be solved for x and y. From equation (32) we get

$$y = \frac{M^2}{x}. \tag{34}$$

Substituting in equation (33), we obtain the quadratic equation

$$x^2 - (M^2 + N^2 + 1)\, x = -M^2, \tag{35}$$

whose solution is

$$x = \frac{M^2 + N^2 + 1 \pm \sqrt{(M^2 + N^2 + 1)^2 - 4M^2}}{2}. \tag{36}$$

Substituting this value of x in equation (32), we have

$$y = \frac{2M^2}{M^2 + N^2 + 1 \pm \sqrt{(M^2 + N^2 + 1)^2 - 4M^2}}. \tag{37}$$

Returning to equations (30) and (31), we get

$$\cosh a_q S_d = \sqrt{x}, \tag{38}$$

and

$$\cos v_q S_d = \sqrt{y}. \tag{39}$$

from which $a_q S_d$ and $v_q S_d$ may be obtained by the use of hyperbolic and trigonometric tables, respectively.

If the spacing distance S_d between the loading coils is known, the values of the attenuation constant a_q, the wave-length constant v_q, and propagation constant p_q which is their vector sum, may all be very easily calculated.

112. The Uniform Line, Nominal to the Regularly Series-Loaded Line. — If the spacing distance of the loading coils is small, the loading-coil impedance may be assumed uniformly distributed over the line. We assume in this manner a smooth line, but one that is only approximately equivalent to the loaded line, because the effect of lumping the loading coils is neglected. Such a line properly may be called *nominal to the loaded line*, the degree of equivalence between it and the loaded line it is supposed to represent being determined by the spacing distance between the loading coils of the latter.

Let us denote by R_l and L_l, respectively, the resistance and inductance of the loading coils affecting the section of length S_d.

Assuming that this resistance and inductance are uniformly distributed over the length of the section, the additional inductive impedance per mile of line is

$$\frac{2\,Z_l\,\epsilon^{j\theta_l}}{S_d} = \frac{R_l + jL_l\omega}{S_d}. \tag{40}$$

Hence the characteristic impedance of the nominal line may be written

$$Z_{N_0} = \sqrt{\frac{R_0 + jL_0\omega + \dfrac{R_l + jL_l\omega}{S_d}}{G_0 + jC_0\omega}}, \tag{41}$$

where the subscript N refers to the nominal line.

The ratio of the characteristic impedance of the smooth line nominal to the loaded line, to the characteristic impedance of the line before loading is

$$\frac{Z_{N_0}}{Z_0} = \sqrt{\frac{R_0 + jL_0\omega + \dfrac{R_l + jL_l\omega}{S_d}}{R_0 + jL_0\omega}},$$

whence

$$Z_{N_0} = Z_0\sqrt{1 + \frac{2\,Z_l}{(R_0 + jL_0\omega)\,S_d}}.$$

Assigning respective arguments to the vectors involved, we get

$$Z_{N_0}\,\epsilon^{j\alpha_N} = Z_0\,\epsilon^{j\alpha}\sqrt{1 + \frac{2\,Z_l}{S_d\sqrt{R_0^2 + L_0^2\,\omega}}\,\epsilon^{\frac{j(\theta_l - \alpha_l)}{2}}}, \tag{42}$$

which gives the characteristic impedance of the smooth line nominal to the loaded line in terms of the loading coil impedance and the constants pertaining to the line before loading.

113. Propagation Constant of the Line Nominal to the Loaded Line. — Referring to equation (40), the propagation constant of the smooth line, nominal to the loaded line, is

$$p_N = \sqrt{\left\{(R_0 + jL_0\omega) + \frac{R_l + jL_l\omega}{S_d}\right\}(G_0 + jC_0\omega)}. \tag{43}$$

The ratio of the propagation constant of the smooth line nominal to the loaded line, to the propagation constant of the line before loading is

$$\frac{p_N}{p_0} = \sqrt{\frac{R_0 + jL_0\omega + \dfrac{R_l + jL_l\omega}{S_d}}{R_0 + jL_0\omega}}. \tag{44}$$

Assigning respective arguments to the vectors involved, we have

$$(45)\quad p_N\,\epsilon^{j\delta_N} = p_0\,\epsilon^{j\delta}\sqrt{1+\frac{2\,Z_l}{S_d\,\sqrt{R_0^2+L_0^2\omega^2}}\,\epsilon^{j(\theta_l-\alpha_1)}}.$$

The wave-length constant of the smooth line nominal to the loaded line, and by which the spacing distance S_d is calculated in § 116, is approximately.

$$(46)\quad v_N = p_0\sqrt{1+\frac{2\,Z_l}{S_d\,\sqrt{R_0^2+L_0^2\omega^2}}}\,\sin\left(\delta+\frac{\theta_l-\alpha_1}{2}\right).$$

The error in assigning to the whole of the expression under the radical of (45) the argument of the second term under the radical, as is done in equation (46), affects the value of S_d computed as in § 116 less than one per cent for the open-wire aerial line with the largest commonly used conductors, and about ten per cent for cables with the smallest commonly used conductors. [1]

114. Degree of Equivalence between the Smooth Line Nominal to the Loaded Line and the Loaded Line. — The smooth line nominal to the loaded line can never be a true equivalent of the loaded line itself, because the effect of lumping the loading coils is neglected. Nevertheless with a spacing distance not exceeding a definite maximum, the nominal smooth line will approach such a close degree of equivalence to the loaded line, that its behavior as a vehicle of telephonic waves is very nearly equal to that of the loaded line it is supposed to represent.

Figure 67 represents the nominal ***T*** circuit of the uniform line nominal to the loaded line, in which, by equations (7) and (8), Chapter XI,

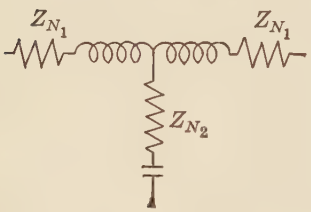

FIG. 67

$$(47)\quad Z_1 = Z_{N_0}\,\frac{p_N S_d}{2}$$

and

$$(48)\quad Z_2 = Z_{N_0}\,\frac{1}{p_N S_d},$$

where Z_1 represents in a concentrated form, one-half of the total inductive impedance of the section of length S_d, with the loading-coil impedance assumed uniformly distributed over it, and where Z_2 denotes the dielectric impedance of the section.

[1] These values are arrived at by calculation in the case of the most unfavorable conditions.

If the loading-coil impedance $2Z_l$ is large in comparison with the inductive impedance $S_d\sqrt{R_0^2 + L_0^2\omega^2}$ of the section before loading, then the nominal T circuit of the smooth line *nominal* to the loaded line would approximate very nearly the true equivalent T circuit of the loaded line, since most of the inductive impedance is in lumped form in the loaded line. On the other hand, if the diagrammatic T circuit of Fig. 67 were to represent the *equivalent* T circuit of the nominal line, it would be the *nominal* T circuit of the loaded line, because the effect of lumping the impedance coils is neglected in the *nominal* line. It follows, therefore, that the regularly series-loaded line approaches equivalence to its nominal smooth line as nearly as the nominal T circuit of the *nominal* line is to the equivalent T circuit of the nominal line, *i.e.*, as nearly as

$$\frac{p_N S_d}{2} \quad \text{approaches} \quad \tanh\frac{p_N S_d}{2}. \tag{49}$$

For, by equation (21), Chapter XI, the value Z_1 for the equivalent T circuit is

$$Z_1 = \tanh\frac{p_N S_d}{2}.$$

Since we have

$$\frac{p_N S_d}{2} = \frac{(a_N + jv_N)}{2} S_d,$$

in the case where the wave-length constant is large compared to the attenuation constant, we have approximately

$$\tanh\frac{p_N S_d}{2} = \tanh j\frac{v_N S_d}{2} = j\tan\frac{v_N S_d}{2}. \tag{50}$$

Under the same assumption, we find

$$\frac{p_N S_d}{2} = j\frac{v_N S_d}{2}. \tag{51}$$

With the same assumption, we see, by (49), (50), and (51), that the regularly series-loaded line approaches equivalence to its smooth nominal line as nearly as

$$\frac{v_N S_d}{2} \quad \text{approaches} \quad \tan\frac{v_N S_d}{2}. \tag{52}$$

Since the wave-length constant v is the angle (in radians) by which the voltage and the current will drop in phase behind their respective sending-end values in one mile of travel, $v_N S_d$ will be the angle

by which the current and the voltage will drop in phase behind their sending-end values in traveling a distance equal to the spacing distance between loading coils. Let us call $v_N S_d = \phi_d$ the ***angular spacing distance*** between loading coils. Then we may say, by (52), that the regularly series-loaded line approaches equivalence to its nominal uniform line as nearly as one-half the *angular spacing distance* between the loading coils approaches numerically its tangent, *i.e.*, as nearly as

$$\frac{\phi_d}{2} \text{ approaches } \tan \frac{\phi_d}{2}.$$

If N_N represents the number of loading coils (of total impedance $2\,Z_l$ each) per wave-length for one definite frequency for the line nominal to the loaded line, then

$$N_N S_d = \lambda_N.$$

Since

$$\lambda_N = \frac{2\,\pi}{v_N},$$

where λ_N is the wave-length corresponding to the frequency under consideration, it follows that

$$v_N S_d = \frac{2\,\pi}{N_N},$$

or

$$\frac{v_N S_d}{2} = \frac{\pi}{N_N}. \tag{53}$$

Hence we may say, by (52), that the regularly series-loaded line approaches equivalence to its nominal smooth line, as nearly as

$$\frac{\pi}{N_N} \text{ approaches } \tan \frac{\pi}{N_N}. \tag{54}$$

115. Minimum Number of Loading Coils per Wave-Length. Maximum Spacing-Distance between Loading Coils. — Equation (15) shows that

$$\cosh p_q S_d = \cosh p_0 S_d + \frac{Z_l}{Z_d} \sinh p_0 S_d.$$

This enables us to determine the minimum number of loading coils per wave-length, that is, the maximum spacing distance between them. Consider a loaded section of an infinitely long regularly series-loaded line. If I_s denotes the current entering the

section, and I_r denotes the current leaving it, we have, for the uniform line equivalent to the loaded line,

$$(55) \qquad \frac{I_s}{I_r} = \epsilon^{p_q S_d} = \cosh p_q S_d + \sinh p_q S_d$$

$$= \cosh (a_q + jv_q) S_d + \sinh (a_q + jv_q) S_d.$$

If we again assume the wave-length constant is large compared to the attenuation constant, this ratio becomes

$$(56) \qquad \frac{I_s}{I_r} = \cosh jv_q S_d + \sinh jv_q S_d = \cos v_q S_d + j \sin v_q S_d.$$

Since the attenuation was neglected in obtaining this equation, the ratio between the current values at the ends of the section is unity. If the ratio between the maximum instantaneous values is considered, that ratio is ± 1, depending upon whether I_s and I_r are in phase or in opposite phase at the same instant of time. It follows from (56) that $\cos v_q S = \pm 1$, which is true when

$$(57) \qquad v_q S_d = n\pi,$$

where n is any integer. Since the wave-length of the uniform line equivalent to the loaded line is

$$\lambda_q = \frac{2\pi}{v_q},$$

it follows from equation (57), that

$$\frac{2\pi}{\lambda_q} S_d = n\pi,$$

or

$$(58) \qquad S_d = \frac{n\lambda_q}{2}.$$

This equation gives the spacing distance in terms of the wave-length.

For $n = 1$, the spacing distance between the loading coils is one-half the wave-length, *i.e.*, the angular distance between the loading coils is π radians. In other words, there must be at least two loading coils to the wave-length.

Let us suppose that the smooth line nominal to the loaded line is used in the determination of the spacing distance between the loading-coils, and the effect of lumping the loading-coil impedance is neglected. Since the ratio of the effect of a distributed inductance

to the effect of an equal lumped inductance is $\pi/2$ (see (27), Chapter VI), it follows that

$$S_d = \frac{n\lambda_N}{\pi}, \tag{59}$$

where λ_N is the wave-length of any harmonic traveling along a smooth line nominal to the loaded line. For $n = 1$, the spacing distance between the coils, in terms of the smooth line nominal to the loaded line, is 2 radians or π coils per wave-length.

Let λ_N be the wave-length of the highest-frequency harmonic encountered in telephonic transmission. If the maximum distance between loading-coils is fixed in terms of the smallest wave-length, the number of loading-coils per wave-length of harmonics of lower frequencies will be larger the lower the frequency. Taking 2400 cycles per second as the highest frequency necessary to successful voice transmission, and assuming a minimum of π loading coils per wave-length at this frequency, we see that there will be a minimum of $3\,\pi$ coils, or approximately 9 coils, per wave-length at the average telephonic frequency of 796 cycles per second.

From this result and from equation (54), under the assumption of negligible attenuation, the degree of equivalence between the actually loaded line, and the smooth line nominal to it, with a maximum spacing distance of 9 coils per nominal wave-length at the average telephonic frequency, is seen to be as

$$\frac{\pi}{9} \quad \text{is to} \quad \tan\frac{\pi}{9} \tag{60}$$

i.e., the smooth nominal line differs from the actual loaded line by 4.4 per cent.

116. Calculation of the Spacing Distance between Loading Coils. — Following the results obtained above, the spacing distance between loading coils may be taken as one-ninth of the wave-length corresponding to a frequency of 796 cycles for the uniform line obtained by assuming that the loading coil impedance is uniformly distributed over the line. The spacing distance S_d may now be determined approximately from a consideration of equations (46) and (53)

$$v_N = p_0 \sqrt{1 + \frac{2\,Z_l}{S_d\sqrt{R_0^2 + L_0^2\omega^2}}}\,\sin\left(\delta + \frac{\theta_l - \alpha_1}{2}\right),$$

$$\frac{v_N S_d}{2} = \frac{\pi}{N_N},$$

where the number of loading coils per wave length at average telephonic frequency is $N_N = 9$. Substituting this value of N_N in the last expression above, and solving for v_N, we obtain

$$v_N = \frac{2\pi}{9 S_d}. \tag{61}$$

Comparing with equation (46), we get

$$\frac{2\pi}{9 S_d} = p_0 \sqrt{1 + \frac{2 Z_l}{S_d \sqrt{R_0^2 + L_0^2\omega^2}}} \sin\left(\delta + \frac{\theta_l - \alpha_1}{2}\right), \tag{62}$$

an equation which is to be solved for S_d. If we set for brevity

$$\sqrt{R_0^2 + L_0^2\omega^2} = z, \tag{63}$$

and

$$p_0 \sin\left(\delta + \frac{\theta_l - \alpha_1}{2}\right) = A, \tag{64}$$

equation (62) may be written in the following simplified form

$$\frac{.698}{S_d} = A\sqrt{1 + \frac{2 Z_l}{z S_d}}.$$

Squaring both sides and rearranging, we find the quadratic equation

$$S_d^2 + \frac{2 Z_l}{z} S_d = \frac{.487}{A^2},$$

whose solution is

$$S_d = -\frac{Z_l}{z} \pm \sqrt{\frac{.487}{A^2} + \frac{Z_l^2}{z^2}}.$$

Substituting for z and A, their values from (63) and (64), we get

$$S_d = -\frac{Z_l}{\sqrt{R_0^2 + L_0^2\omega^2}} \pm \sqrt{\frac{.487}{\left[p_0 \sin\left(\delta + \frac{\theta_l - \alpha_1}{2}\right)\right]^2} + \frac{Z_l^2}{R_0^2 + L_0^2\omega^2}}, \tag{65}$$

the positive value being, obviously, the only real solution.

117. Calculation of the Transmission Constants of the Line Nominal to the No. 10 B. & S. Regularly Series-Loaded Line. — Let us suppose that the No. 10 B. & S. aerial line is a regularly series-loaded line with double coil loading impedances of total combined inductance of .2 henries, and combined effective resist-

ance of 6.5 ohms at the average telephonic frequency corresponding to $\omega = 5000$ radians per second. Then the total impedance of the double loading coil is

$$2 Z_l \epsilon^{j\theta_l} = 1000 \epsilon^{j89.62°},$$

and the impedance per coil is, therefore,

$$Z_l \epsilon^{j\theta_l} = 500 \epsilon^{j89.62°}.$$

The inductive impedance per mile, by previous calculation (Chapter IV), is

$$\sqrt{R_0^2 + L_0^2\omega^2}\, \epsilon^{j\alpha_1} = 21.1 \epsilon^{j60.7°}$$

We have also

$$p_0 \epsilon^{j\delta} = .0292 \epsilon^{j71.8°}.$$

Substituting these values in equation (65) and performing the required calculations, we find that the spacing distance between loading coils is

$$S_d = -23.7 + 33.34 = 9.64 \text{ miles}.$$

Substituting this value of S_d in (45), we get

$$p_N \epsilon^{j\delta_N} = .0292 \epsilon^{j71.8°} \sqrt{1 + \frac{1000 \epsilon^{j89.62°}}{9.64 \times 21.1 \epsilon^{j60.7°}}}$$
$$= .0705 \epsilon^{j77.5°}.$$

The wave-length constant of the smooth line nominal to the loaded line is

$$v_N = p_N \sin \delta_N = .0705 \sin 77.5° = .0688$$

The attenuation constant is

$$a_N = p_N \cos \delta_N = .0705 \cos 77.5° = .0154$$

The wave-length is

$$\lambda_N = \frac{2\pi}{.0688} = 91.2 \quad \text{miles}$$

and the velocity of propagation is

$$V_N = \frac{\omega}{v_N} = \frac{5000}{.0688} = 72670 \quad \text{miles per second.}$$

The characteristic impedance of the smooth line nominal to the loaded line, by (42), is

$$Z_{N_0} \epsilon^{j\alpha_N} = 723 \epsilon^{-j11.1°} \sqrt{1 + \frac{1000 \epsilon^{j89.62°}}{9.64 \times 21.1 \epsilon^{j60.7°}}} = 1755 \epsilon^{-j1.64°}.$$

It should be noted that by loading the No. 10 B. & S. line, the characteristic impedance has been more than doubled, while the characteristic phase angle is greatly reduced. This indicates that distortion ha been dec eased considerably.

118. Calculation of the Transmission Constants of the Uniform Line Equivalent to the Regularly Series-Loaded No. 10 B. & S. Line. — With the spacing distance determined by means of the smooth line nominal to the loaded line, the transmission constants of the actually lumped loaded line may be calculated by means of the equations developed in §§ 109–111. Thus the characteristic impedance of the loaded No. 10 B. & S. line, by (18), is

$$Z_{q0}\epsilon^{j\alpha_q} = 723\,\epsilon^{-j11.1^\circ}\sqrt{1+\left[\frac{500\,\epsilon^{j89.62^\circ}}{723\,\epsilon^{-j11.1^\circ}}\right]^2+\frac{2\times 500\,\epsilon^{j89.62^\circ}}{723\,\epsilon^{-j11.1}\tanh\,p_0S_d}},$$

where

$$p_0S_d = (a_0 + jv_0)\,S_d = (.009119 + .02774\,j)\,9.64 = .088 + .267\,j,$$

whence $a_0S_d = .088$, and $v_0S_d = .267$; it follows that

$$\sinh p_0S_d = \sqrt{\cosh^2 a_0S_d - \cos^2 v_0S_d}\;\epsilon^{j\tan^{-1}[\tan v_0S_d\,/\,\tanh a_0S_d]} = .275\,\epsilon^{j72.1^\circ}.$$

Moreover,

$$\cosh p_0S_d = \sqrt{\cosh^2 a_0S_d - \sin^2 v_0S_d}\;\epsilon^{j\tan^{-1}[\tanh a_0S_d\tan v_0S_d]} = .9692\,\epsilon^{j1.37^\circ};$$

hence

$$\tanh p_0S_s = \frac{\sinh p_0S_d}{\cosh p_0S_d} = \frac{.275\,\epsilon^{j72.1^\circ}}{.9692\,\epsilon^{j1.37^\circ}} = .2837\,\epsilon^{j70.73^\circ}.$$

Substituting this value of $\tanh p_0S_d$ in the expression for Z_{q0}, we get

$$Z_{q0}\,\epsilon^{j\alpha_q} = 723\,\epsilon^{-j11.1^\circ}\sqrt{1 + .4782\,\epsilon^{j201.44^\circ} + 4.877\,\epsilon^{j30^\circ}},$$

but we have

$$.4782\,\epsilon^{j201.44^\circ} = -.4469 - j\,.1652,$$

and

$$4.877\,\epsilon^{j30^\circ} = 4.222 + j\,2.422.$$

Hence

$$1 + .4782\,\epsilon^{j201.44^\circ} + 4.877\,\epsilon^{j30} = 4.7751 + j\,2.2698 = 5.2375\,\epsilon^{j25.4^\circ}.$$

Substituting in the expression for the characteristic impedance, we get

$$Z_{q0}\,\epsilon^{j\alpha_q} = 723\,\epsilon^{-j11.1^\circ}\sqrt{5.2375\,\epsilon^{j25.4^\circ}} = 1662.5\,\epsilon^{j1.6^\circ}.$$

It will be observed that the characteristic impedance of the loaded line is more than double the characteristic impedance of

the non-loaded line. It is slightly less than the characteristic impedance of the smooth line nominal to the loaded line. The characteristic phase angle is positive as compared with the characteristic phase angle of the line before loading, indicating that slightly more inductance is used than that called for by the condition for distortionles transmission.

To calculate the other transmission constants, equation (15) may be used most effectively as follows:

$$\cosh p_q S_q = \frac{500\ \epsilon^{j89.62^\circ}}{723\ \epsilon^{-j11.1^\circ}} \times .275\ \epsilon^{j72.1^\circ} + .9692\ \epsilon^{j1.37^\circ}$$
$$= .2176\ \epsilon^{j172.82^\circ} + .9692\ \epsilon^{j1.37^\circ} = .75302 + j\ .030536.$$

Hence, by equations (24) and (25)

$$\cosh a_q S_d \cos v_q S_d = M = .75302,$$

and

$$\sinh a_q S_d \sin v_q S_d = N = .030536,$$

whence by equation (36),

$$\cosh a_q S_d =$$
$$\sqrt{\frac{.75302^2 + .030536^2 + 1 \pm \sqrt{(.75302^2 + .030536^2 + 1)^2 - 4 \times .75302^2}}{2}}$$
$$= \sqrt{\frac{1.56797 \pm .43605}{2}}$$

Since the hyperbolic cosine cannot be less than 1, there is only one real solution

$$\cosh a_q S_d = + \sqrt{\frac{1.5697 + .43605}{2}} = 1.001005,$$

and from hyperbolic tables, we find

$$a_q S_d = .04202$$

Using the known value of the spacing distances S_d, we get

$$a_q = \frac{.04202}{9.64} = .004359$$

Moreover, by equation (37), we get

$$\cos v_q S_d = \sqrt{\frac{2 \times .75302^2}{1.56797 + \sqrt{1.56797^2 - 4 \times .75302^2}}} = .7457$$

From trigonometric tables, the angle whose cosine is .7457 is

$$v_q S_d = .8525 \quad \text{radians.}$$

Using the known value of the spacing distance S_d, we get

$$v_q = \frac{.8525}{9.64} = .08843$$

With the value of the wave-length constant obtained above, the wave-length and the velocity of propagation at the average telephonic frequency of 796 cycles per second are calculated as follows:

$$\lambda_q = \frac{2\pi}{.08843} = 71.1 \quad \text{miles},$$

$$V_q = \frac{5000}{.08843} = 56540 \text{ miles per second.}$$

The propagation constant is obtained by taking the vector sum of the attenuation constant and wave-length constants

$$p_q = .004359 + .08843\,j = .08854\,\epsilon^{j87.5°}.$$

119. Calculation of the Transmission Constants of the Ideally Loaded No. 10 B. & S. Line. — The theoretical inductance necessary for the distortionless transmission of telephonic waves is

$$L_i = \frac{R_0 C_0}{G_0},$$

which for the No. 10 B. & S. line amounts to

$$L_i = \frac{10.33 \times 80.18 \times 10^{-10}}{5 \times 10^{-6}} = .016565 \quad \text{henries.}$$

Since the line has a natural inductance of .003676 henries per mile, the additional inductance per mile of line for distortionless transmission must be

$$.016565 - .003676 = .012889 \quad \text{henries.}$$

If this is distributed uniformly over the line, per mile of line, the impedance of the thus ideally and uniformly loaded line would be

$$Z_{i0} = \sqrt{\frac{L_i}{C_0}} = \sqrt{\frac{.016565}{80.18 \times 10^{-10}}} = \sqrt{\frac{R_0}{G_0}} = \sqrt{\frac{10.33}{5 \times 10^{-6}}} = 1437\,\epsilon^{j0°},$$

which is the natural or surge impedance of the loaded line.

The attenuation constant of the ideally loaded line by equation (19), Chapter XII, is

$$a_i = \sqrt{R_0 G_0} = \sqrt{10.33 \times 5 \times 10^{-6}} = .00717,$$

and the wave-length constant by equation (20), Chapter XII, is

$$v_i = \omega\sqrt{L_i C_0} = .05762,$$

whence the wave-length for the average frequency of 796 cycles per second is

$$\lambda_l = \frac{2\pi}{.05762} = 109 \quad \text{miles},$$

and the velocity of propagation is

$$V_a = \frac{5000}{.05762} = 88512 \quad \text{miles per second.}$$

The propagation constant of the ideally loaded line is found by taking the vector sum of the attenuation constant and the wave-length constant:

$$p_i = 00717 + j\ .05762 = 05806\ \epsilon^{j82.9°}.$$

The following table gives a summary of the results obtained for the No. 10 B. & S. line.

TABLE 9

TRANSMISSION CONSTANTS	NON-LOADED LINE	IDEALLY LOADED	$2 Z_l \epsilon^{j\theta l} = 1000\ \epsilon^{j89.5°};\ S_d = 9.64$	
			UNIFORM LOADING	REGULAR SERIES LOADING
Z_0	$723\ \epsilon^{-j11.1°}$	$1437\ \epsilon^{j0°}$	$1755\ \epsilon^{-j1.64°}$	$1662\ \epsilon^{j1.6°}$
p	$.0292\ \epsilon^{j71.8°}$	$.05806\ \epsilon^{j82.9°}$	$.0705\ \epsilon^{j77.5°}$	$.08854\ \epsilon^{j87.5°}$
a	.009119	.00717	.0154	.004359
v	.02774	.05762	.0688	.08843
λ	226.4	109	91.2	71.1
V	180245	88512	72670	56540

120. Effect of Artificial Loading on Energy Transfer. — The prime object of inductively loading a line is to bring about distortionless transmission, by making the characteristic impedance of the line and the propagation factor independent of the frequency. This forces all components of the complex telephonic wave to attenuate at the same rate, and to propagate with the same velocity, thereby maintaining the phase relationship with respect to one another constant.

With ideal loading, the line phase angle α is reduced to zero. With series lump loading, the phase angle is reduced to a sufficiently small value so that the character of the electrical counterpart of the voice wave is maintained more or less intact during the period of propagation from one line terminal to the other.

While the argument of the characteristic impedance is reduced, the modulus, *i.e.*, the numerical value of the characteristic impedance, is greatly increased. Under similar conditions with respect to length of line and receiving-end impedance, the sending-end impedance of the loaded line will be larger than the sending-end impedance of the line before loading. Hence with equal sending-end voltage the sending-end current will be smaller with loaded lines. The sending-end voltamperes consequently will be less with the loaded line, an obvious disadvantageous effect of loading.

However, since the phase angle is very small, the line power-factor is nearly unity. The energy delivered to the loaded line, though less than that obtainable with the line before loading, is transferred with a smaller reactive component, for by loading, the inductive line reactance is made very nearly equal to the dielectric reactance of the line, so that each is capable of nearly complete discharge through the other. The successive conversions of energy from the electric to the magnetic field and vice-versa are nearly complete; very little energy is left stored along the line. Consequently more dissipative energy is transferred. It follows, therefore, that while the energy delivered to the loaded line is less than that delivered to a non-loaded line of the same size conductor and spacing between conductors and under similar end conditions, it is almost wholly dissipative in character. Moreover, since the attenuation is smaller for the loaded line, the line loss occurring during propagation will be decidedly less.

The energy gain from the reduction in the reactive component, and the gain resulting from the decrease in line dissipative loss, becomes apparent at and beyond a point along the line, at which the loss in energy due to the smaller sending-end current is fully balanced or compensated for by the gains mentioned above.

If E_s = sending-end voltage for loaded and non-loaded line,
Z_s = sending-end impedance of long line before loading,
Z_{sl} = sending-end impedance of the same line after loading,
a_0 = attenuation constant of line before loading,
a_l = attenuation constant of line after loading,

then

$$\frac{E_s}{Z_s} = I_s = \text{sending-end current of line before loading,}$$

and

$$\frac{E_s}{Z_{sl}} = I_{sl} = \text{sending-end current of line after loading,}$$

whence

$E_s I_s$ = sending-end voltamperes of line before loading, and
$E_s I_{sl}$ = sending-end voltamperes of line after loading.

If S is the distance from the sending end at which the gain in energy balances the loss due to the higher sending-end impedance, then the voltamperes for the line before loading must equal the voltamperes for the line after loading at that distance from the sending end, *i.e.*,

$$E_s I_s \epsilon^{-2a_0 S} = E_s I_{sl} \epsilon^{-2a_l S},$$

whence

$$2\, a_l S \log_{10} \epsilon + \log_{10} I_{sl} = 2\, a_0 S \log_{10} \epsilon + \log_{10} I_s$$

Solving for S, we get

$$S = \frac{\log_{10} \dfrac{I_s}{I_{sl}}}{2\,(a_l - a_0) \log_{10} \epsilon} = \frac{\log_{10} \dfrac{Z_{sl}}{Z_s}}{2\,(a_l - a_0) \log_{10} \epsilon}.$$

It follows that the gain in energy resulting from loading becomes available beyond this distance, as measured from the sending end of the line.

For our No. 10 B. & S. line, regularly series-loaded, and assumed very long, so that the sending-end impedance is sensibly equal to the characteristic line, impedance before and after loading, respectively,

$$S = \frac{\log_{10} \dfrac{1662}{723}}{2\,(.009119 - .004349)\ .434249} = 87.26 \quad \text{miles.}$$

This indicates that up to a distance of 87.26 miles from the sending end, the voltamperes at any point along the loaded line will be smaller than the voltamperes at corresponding points along the non-loaded line, while from that distance onward, the voltamperes at any point along the loaded line will be larger than at corresponding points of the non-loaded line.

Considering only current values, there is no gain in current for a loaded No. 10 B. & S. line shorter than $2 \times 87.26 = 174.52$ miles; and for longer lines the gain is available only beyond this distance from the sending end.

121. Use of Terminal Repeating Coils for the Apparent Equilization of Impedances. — The higher sending-end impedances resulting from inductively loading a line would necessarily demand a

transmitting apparatus of higher voltage, which would not be suitable for use with non-loaded lines. This would imply different sets of transmitting apparatus for the two types of lines, a very decided disadvantage in practical telephony.

The necessary higher voltage for the loaded line may, however, be obtained by means of step-up transformers, as is done in heavy power transmission. While both object and method are identical in the two cases, in telephone phraseology, the function of the transformer, referred to as a repeating coil, is thought of as changing the impedance, rather than as altering the voltage and current values, and for the very natural reason that it is impossible to rate telephone apparatus in terms of volts and amperes. It must be clearly understood, however, that a step-up repeating coil does not actually decrease the line impedance. With the low-voltage coil across the transmitting apparatus, and the high-voltage coil across the high sending-end impedance of the loaded line, the transformer permits the transmitting apparatus to work at low voltage and comparatively high currents, which is equivalent to a reduction in impedance.

We must also bear in mind that, since a telephone line carries waves back and forth, as the speakers at its terminals converse with each other, *i.e.*, since the terminals of the line are intermittently sending and receiving ends, repeating coils must be installed at both ends of the line, functioning either as step-up or as step-down transformers, depending upon whether the terminals are, for the time being, sending or receiving ends, respectively.

To make this idea clearer, consider a loaded line of characteristic impedance Z_{0l}. In order that this line behave like an infinitely long loaded line, the receiving-end impedance must be equal

Fig. 68

to Z_{0l}. In that case, the sending-end impedance will also be equal to Z_{0l}. If the transmitting and receiving apparatus of this loaded line are to be same as the ones used for a non-loaded line, *i.e.*, if the sending-end voltage and receiver impedance should be the

same, respectively, for the two lines, the transmitter voltage must operate against a sending-end impedance equal to that of the non-loaded line. In other words, the sending-end impedance Z_{0l} of the loaded line must be effectively reduced from Z_{0l} to Z_0.

To accomplish this, the loaded line is connected to the terminal apparatus through a step-up repeating coil, with the high voltage coil on the line side, as shown diagrammatically in Fig. 68.

Let us assume that the leakage reactance in the repeating coil is negligible. Then if

E_s = voltage of the transmitting apparatus,

Z_0 = sending-end impedance against which transmitter voltage is to operate,

$\frac{E_s}{Z_0} = I_s$ = transmitter current output in primary of repeating coil,

n = ratio of transformation of repeating coil,

the secondary voltage is

$$E_{sl} = n E_s,$$

and the current at the sending end of the loaded line is

$$I_{sl} = \frac{I_s}{n}.$$

But since $E_{sl} = I_{sl} Z_{0l}$, it follows that

$$nE_s = \frac{I_s}{n} Z_{0l},$$

or

$$\frac{E_s}{I_s} = \frac{Z_{0l}}{n^2} = Z_0,$$

whence

$$n = \sqrt{\frac{Z_{0l}}{Z_0}}. \tag{66}$$

This result shows that to apparently reduce the sending-end impedance of the loaded line to that of the line before loading, the ratio of transformation of the repeating coil used for the purpose must be equal to the square root of the ratio of the impedances.

The preceding discussion shows that repeating coils enable us to use the same terminal apparatus with loaded lines having high characteristic impedance as with non-loaded lines of lower characteristic impedance. Applying the same train of reasoning, it

could be easily seen that through the use of repeating coils, standard terminal apparatus may be, and are, generally adapted to be used in connection with lines having any characteristic impedance whatever. Evidently the ratio of transformation of the repeating coil must be equal to the square root of the ratio of the line impedance, whatever it may be, to the impedance of the standard apparatus, whether transmitter or receiver.

Another useful application of repeating coils which is based on the results of the preceding discussion, is to equalize the impedances of two or more lines of different transmission constants, when connected into what is called a *composite line*. Thus two lines of characteristic impedances Z_{01} and Z_{02}, when connected into a single line, must be connected through repeating coils with a ratio or transformation

$$n = \sqrt{\frac{Z_{01}}{Z_{02}}}$$

with the high-voltage coil on the side of the high-impedance line.

122. Loading of Underground Cable Lines. — The essential differences between the properties of an underground cable line and those of open aerial lines are: (*a*) The open line conductors, being subject to varying weather conditions, must necessarily be of a larger size than the cable conductors, hence the ohmic resistance per mile of cable line is greater than the resistance per mile of aerial line. (*b*) The cable conductors, being quite close to each other and of smaller diameter, and the insulation between them being of a higher specific inductive capacity than air, the capacitance per mile of cable line is larger than that of the aerial per mile of line. (*c*) Since the interaxial distance between the cable conductors is small, the inductance per mile of cable line is negligibly small. (*d*) Since the cable conductors are well insulated, leakage conductance is almost a negligible quantity, while with the aerial line it is not only considerable in amount, but subject to wide variations, due to changes in weather conditions.

Keeping in mind these differences in the properties of cable and aerial telephone transmission lines with reference to the necessary conditions for distortionless transmission $LG = RC$, it is clear that aerial lines will transmit telephonic waves more effectively than cable lines, to which obviously the principle of artificial loading has found its widest application.

The small value of LG in comparison with RC evidently calls for a much heavier loading for cable lines, with regard to the inductance of the loading coils, as well as a smaller spacing distance between them. For cable lines, the loading coils often have an inductance of .251 henries each, and are spaced about 1.14 miles apart, while for aerial lines, the inductance per loading coil is .246 henries, and the spacing distance is about 7.88 miles.

Not only has artificial loading of cable lines greatly improved underground telephone transmission, but it has made possible the placing of many long distance lines underground, thereby securing the benefits of continuity of service free from interruptions due to storms, and also the lowering of the maintenance costs.

CHAPTER XIV

AMPLIFIERS

123. Amplification, a Necessity for Long-Distance Telephony. — The discussion given in the preceding chapter disclosed the fact that distortionless transmission obtained by inductively loading a line results also in a decrease of the line attenuation constant, and the consequent gain in the distance over which articulated speech may be transmitted.

In order to extend indefinitely the range of transmission, however, it is imperative that telephonic waves should be re-energized, or reinforced, from one or more local sources along the line, to compensate for the energy loss in line resistance and leakage, and thus permit transmission to more distant points. This re-energizing process is called *amplification*, and the apparatus used for the purpose are called *amplifiers*, or more commonly, *repeaters*.

In a more general sense, amplifiers act as *relays*, inasmuch as the attenuated energy constituting the telephonic train of waves, emanating from the sending terminal of the line, merely controls the energy supplied from the local source comprised within the amplifying circuit. The re-energized train of waves emitted from the amplifier is thereby made a faithful reproduction of the attenuated and controlling train of waves, with respect to frequency and phase relationship, so that when reconverted into sound, through the agency of the receiver, the quality of the sounds remains unimpaired.

124. The Shreeve Repeater.[1] — Prior to the introduction of the three-electrode thermionic tube in 1913, an amplifying device of the microphonic type received considerable attention, and gave more or less satisfactory results. The *Shreeve repeater*, so called after its inventor, consisted essentially of a sensitive microphone attached mechanically to a telephone receiver. Obviously, in this

[1] For a more complete description of the Shreeve repeater, see a paper entitled *Telephone Repeaters*, presented by B. Gherardi and Frank B. Jewett before the A. I. E. E. in 1919.

device we have a true *repeater action*, since the energy of the incoming train of waves is converted into mechanical energy, resulting in vibrations of the receiver diaphragm which control the electric energy from the local source by the variations of the microphone resistance.

The inherent defect of this repeater is clearly in the vibrating element, which necessarily must be actuated by exceedingly small variations of voltage and current, characteristic of the electrical replica of voice waves. Hence, its sensitiveness is a direct function of the controlling energy. It will respond more readily to comparatively large variations in energy, and may fail to respond when most needed, when the energy of the controlling wave is small.

125. Fundamentals of Thermionic Emission; the Thermionic Tube. — The success of the three-electrode thermionic vacuum tube as an amplifier lies in the absence of any mechanically vibrating parts which would require comparatively large amounts of energy to operate them. While it is outside the scope of this book to discuss in any detail the physical phenomena pertaining to *thermionics*, a more or less general and rather elementary treatment of the thermionic tube as an amplifier is imperative, in order that the student obtain an understanding of its application to long distance telephony.[1]

The name *thermionics* is given to that branch of the physical sciences which deals with electron emission from hot metallic filaments in vacuo. The explanation of the phenomena of electronic emission is based upon the modern physical conception that matter in its ultimate state is an aggregate of certain nuclei and negative charges of electricity, each of which may be regarded as a unit by itself, and which are called *electrons*. The fact that these electrons have been found to be the same for all kinds of matter, points to the very remarkable, and as yet speculative assumption, that the ultimate constituent of all matter is the same. Moreover, since an electric charge is the seat of a definite amount of potential energy, each electron represents a definite amount of potential energy. If this assumption is correct, and there is no evidence to the contrary, it indicates the very interesting and rather remarkable possibility that matter in its ultimate form is *energy*.

[1] For a comprehensive treatment of the subject of thermionics, the reader is referred to *Principles of Radio Communication*, by J. H. Morecroft, and to *The Thermionic Vacuum Tube*, by H. J. Van Der Bijl.

Since negative charges of electricity exert a repelling action upon one another, there is reason to believe that the electrons, being equal negative charges, repel one another, and consequently are in a continuous state of motion, though bound to the space in which they move, by a force of attraction. This leads to another interesting fact, namely, that what is now conventionally called *positive electricity* is merely a deficiency in normal amount of negative charge, or rather a deficiency in the normal number of electrons. The affinity or attractive force between positive and negative charges of electricity may be thought of merely as space affinity for electrons. Under favorable conditions, free electrons will be attracted and move towards space void of electrons or to matter deficient in normal number of electrons.

Natural conditions may exist, or artificial conditions may be established under which electrons are forced to escape, especially from metals, and more easily from some as compared with others. Radioactive matter emits them freely. A copious emission of electrons can be obtained from an electrically heated tungsten filament in *vacuo*. A still greater emission can be obtained from a barium or strontium oxide coated platinum filament, similarly enclosed in a highly exhausted glass bulb.

The escape of electrons from hot filaments is attributed to the additional energy imparted to them by heating. This results in their acquiring a higher velocity, which carries them past the surface of the hot body. The escaped electrons, however, by their repelling action, interfere with further escape of others, and a steady state is soon reached, the immediate space surrounding the filament becoming charged with electrons, most of which are concentrated near the emitting surface. Conclusive evidence of this electronic emission may be obtained by placing a metallic plate near the filament. This plate, when tested by means of an electroscope, will show a negative charge of electricity.

It is obvious from what has been said, that if a continuous emission of electrons is to be obtained from a hot filament, some scheme of carrying them away as quickly as they are emitted must be employed. Since, as has been mentioned, electrons will move to matter deficient in normal number of electrons, *i.e.*, to matter conventionally said to be charged with positive electricity, it is evident that all that is necessary to obtain a constant stream of electrons is to place near the electron-emitting hot filament a metallic

plate, and maintain it at a constant potential, and what is conventionally called a *positive potential*, with respect to the filament. This is done by connecting the plate to the positive terminal of a battery and completing the circuit to positive terminal of the filament battery, as shown diagrammatically in Fig. 69.

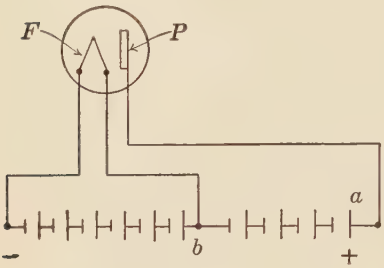

Fig. 69

Under these conditions, a constant stream of electrons emitted from the hot filament passes across the space from filament to plate, where they are absorbed at a definite rate $-dq/dt$, resulting in a current flow in the circuit $P - F - b - a$, in the conventional direction from plate across the intervening space to filament.

That a current flow is obtained under such artificial conditions was accidentally discovered by Thomas Edison in 1883. Since no explanation of the phenomenon could be given at that time, it was, and is still, referred to as the *Edison effect*.

126. The Two-Electrode Thermionic Tube. — The vacuum thermionic tube described in the latter part of the preceding section is called a *two-electrode tube*, and more generally a *Fleming valve*, after Dr. J. A. Fleming of England, who in 1905 conceived its usefulness as a rectification device for radio detection.

It was seen that two necessary conditions must be satisfied in order that a stream of electrons shall proceed from filament to plate, *i.e.*, to obtain a current flow in the plate-filament circuit, to be referred to subsequently as the *plate current*, and denoted by I_p: (1) the filament must be hot, and (2) the plate must be kept at a positive potential with respect to the filament.

The rate of electron emission varies, however, with the temperature of the filament, and the rate of their absorption depends upon the positive potential of the plate. It follows, therefore, that since the plate current depends both upon the rate of electron emission and upon the rate of their absorption, its value is a joint function of filament temperature and plate potential.

The relation obtained experimentally between the plate current and filament temperature T_f with the plate maintained at a fixed

positive potential with respect to filament, is shown by the curve, Fig. 70. This curve shows that as the temperature of the filament is increased, more and more electrons are emitted from its hot surface, and are absorbed by the plate as fast as they are emitted, until a temperature is reached at which the rate of electron emission from the hot filament exceeds the rate of their absorption by the plate. The plate current then reaches a more or less definite value called *saturation current*, for the particular positive potential at which the plate is maintained. Similarly, if the filament temperature is maintained constant, the experimental relationship between plate current and plate potential is as shown in Fig. 71. This curve shows that as the plate potential is increased, thereby increasing its ability to absorb electrons, the plate current increases to a more or less constant value, a state in which the electron absorbing ability of the plate exceeds that of electron emission from the filament. This plate current is similarly called *saturation current* for the particular temperature at which the filament is operated.

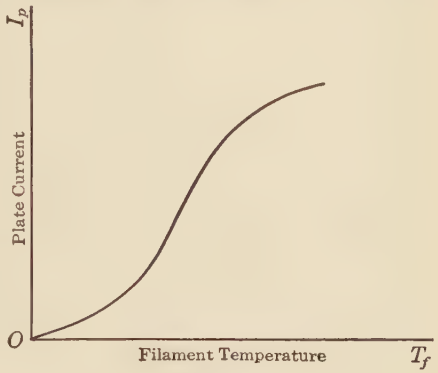

FIG. 70

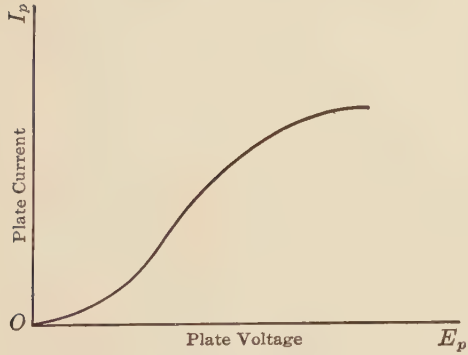

FIG. 71

From what has been said, it should be clear that if the plate potential is negative with respect to the potential of the filament, its ability to absorb electrons disappears. The negatively charged plate repels the electrons emitted by the filament, and the plate current is therefore zero for all values of negative plate potential. A plate current will result then, only

when the plate is at a positive potential with respect to the filament, and will flow only in the conventional direction from the plate to filament. For this reason the thermionic tube is said to have *one-sided* or *unilateral conductivity*. It is this *one-way conductivity* that makes the two-electrode vacuum tube particularly well adapted as a rectification device for alternating currents of any frequency whatever, and as such, useful as a wireless detector under the name of the *Fleming valve*.

The variable slope of the I_p versus E_p curve, Fig. 71, is of the nature of an admittance. For the definite operating temperature of the filament, and plate potential for which the tube has been designed, the ratio

$$\frac{I_p}{E_p} = Y_p$$

is called *internal admittance* of the tube, and its reciprocal $1/Y_p = Z_p$ is the *internal impedance*.

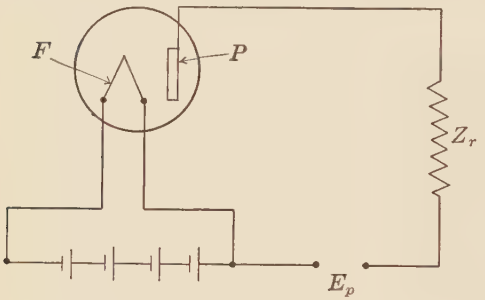

Fig. 72

To gain a better understanding of the characteristic behavior of the tube as a rectifier, consider a two-electrode tube, diagrammatically shown in Fig. 72, and assume that an alternating electromotive force of any frequency f and amplitude $E_p = b$ (Fig. 73) is impressed upon the plate-filament circuit in which a receiver of impedance Z_r is connected in series. As the impressed electromotive force grows from 0 to its negative maximum $-E_p$, and back to zero, describing the negative half cycle $a - b - c$, Fig. 73, the plate being at a varying negative potential to the filament during this length of time, it cannot absorb any electrons, and the plate current in circuit $P - F - Z_r$ is, therefore, zero. When the plate electromotive force reverses and grows to its maximum positive value, $+E_p$, and back to zero, along the positive half cycle $c - d - e$, the plate is at a sinusoidally increasing and decreasing positive potential, and the plate current will increase from zero to a maximum I_{p_1}, and back to zero along the characteristic I_p versus E_p curve of the tube. During the next negative half-

cycle of the electromotive force, the plate current remains again at a zero value. With sinusoidal variations in plate potential, there will result then a unidirectional plate current *pulsating* with a frequency equal to that of the electromotive force producing it.

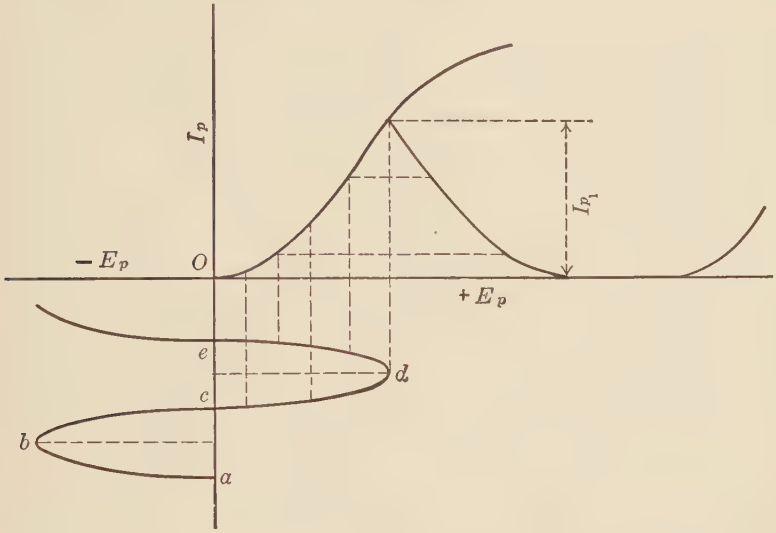

FIG. 73

127. The Three-Electrode Thermionic Tube. — With the filament at constant temperature, and plate maintained at a constant positive potential with respect to the filament, the rate of electron emission by the filament and their rate of absorption by the plate is constant, resulting, as was stated in the preceding section, in a plate current of constant value. If now an intermediary electrode in the form of a wire mesh, commonly referred to as the ***grid,*** is placed between the filament and plate, the value of the plate current is made subject to the potential between this intermediary grid or *third electrode* and the filamant. De Forest in 1907 conceived this method of plate current variations and applied it to his wireless audion detector.

With constant filament temperature and constant positive plate potential, the experimental relationship between plate current and potential between grid and filament (to be referred to subsequently as *grid potential* and denoted by E_g) is as shown in Fig. 74.

An investigation of this curve shows that with the grid potential equal to zero, a rather large proportion of the electrons

emitted from the filament will pass through the grid meshes in their course to the positive plate. Some will fall on the grid, imparting to it a negative charge, which through its repelling action, will reduce the number of electrons that otherwise would land on and be absorbed by the plate. The plate current will accordingly be smaller than if the grid were not there.

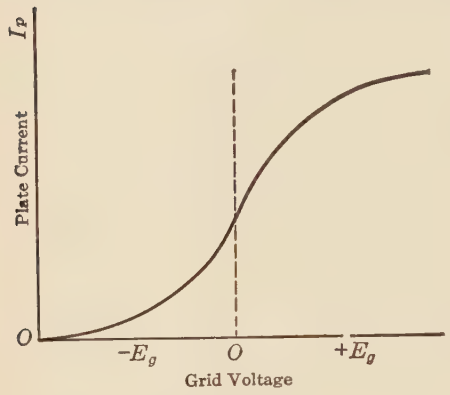

FIG. 74

If the grid is made positive to the filament, it assists the plate, so to speak, in pulling electrons away from the filament; the stream of electrons increases, more pass through the grid meshes to the plate, and the plate current I_p increases. With the grid-filament circuit closed, the grid will also absorb electrons at a definite rate, depending upon its potential with respect to the filament, resulting in a grid current in the conventional direction from grid to filament.

On the other hand, if the grid potential is made negative with respect to the filament, by connecting it to the negative terminal of a battery and completing the circuit to the negative terminal of the filament battery, as shown in Fig. 75, the grid will exert a greater repelling action on the electrons. As its negative potential is increased, fewer and fewer electrons will get through its meshes and land on the positive plate. The plate current is decreased until, at a definite negative potential established between the grid and filament, the repelling action of the grid is too great to permit the passage of any electrons through its meshes, and the plate current is equal to zero.

Since no electrons can be emitted by the grid, no current can flow in the conventional direction from filament to grid. Hence, like the plate-filament circuit, the grid-filament circuit has a unilateral conductivity. We shall see that this property makes the three-electrode tube of great value both as an energy amplification device and as a variable frequency generator or *oscillator*, as it is more frequently called.

Referring to the relationship between plate current and grid potential, Fig. 74, it will be noted that the slope of the curve is of the nature of an admittance

$$\frac{I_p}{E_g} = Y_m,$$

and is called *mutual admittance* of the tube. The reciprocal of the mutual admittance is called *mutual impedance* Z_m. When the grid potential is positive, the mutual impedance is of the nature of an *inductive reactance*, and when the grid potential is negative, it is of the nature of a capacity reactance.

128. Amplifier Action of the Three-Electrode Tube. — For convenience in making connections the highly exhausted glass bulb containing the filament, grid, and plate (more generally two grids

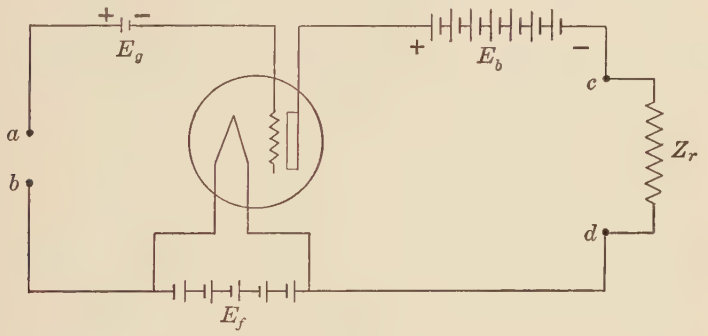

Fig. 75

connected together and two plates also interconnected) is cemented to a brass base that has four terminals insulated from each other. Two of these terminals are connected to the filament, one to the plate and one to the grid. The base fits into a specially made socket provided with four springs corresponding to the above four tube terminals. By means of these, electrical connections are made to the battery which supplies the current for heating the filament, and to the batteries by means of which the required grid and plate potentials are maintained. The connections are shown schematically in Fig. 75.

The grid-filament terminals a and b are called *input terminals*, and the plate-filament terminals c and d are called *output terminals*.

To understand the action of the tube as an amplifier, let us set

E_b = battery voltage in the plate-filament circuit;
Z_p = the internal impedance of the tube;
Z_r = impedance of receiving apparatus connected to the output terminals $c - d$;
$-E_g$ = the negative grid potential of such a magnitude that with any value of electromotive force impressed at the input terminals a and b the potential of the grid-filament is always negative with respect to the plate;
E_{ga} = the amplitude of a sinusoidal electromotive force, of any frequency f superimposed upon the grid potential;
E_{pa} = the amplitude of the alternating component of plate potential.
I_p = the plate current when grid-filament potential is $-E_g$.

E_g and E_p are measured from the grid and plate respectively to some convenient point in the common circuit, such as a point on the filament in Fig. 75.

Under these assumed conditions, the instantaneous grid potential is

$$e_g = -E_g + E_{ga} \sin \omega t,$$

and the limiting values of its variation are $-E_g - E_{ga}$ and $-E_g + E_{ga}$, which are both negative quantities, since by assumption E_{ga} is numerically less than E_g.

As the alternating electromotive force impressed at the input terminals passes through its negative half-cycle along $e - f - g$, Fig. 76, the plate-filament current decreases along the characteristic curve I_p versus E_g, to a minimum value $I_p - I_{pa}$ and back to I_p. When the alternating electromotive force passes through its positive half-cycle along $g - h - i$, the plate current increases along the I_p versus E_g curve from I_p to a maximum $I_p + I_{pa}$ and back to I_p. A true sinusoidal variation of plate current for sinusoidal variations of superimposed grid potential would obviously be obtained if the portion of the characteristic I_p versus E_g curve, along which the plate current varies, were a straight line. Ordinarily this may be obtained by an adjustment of resistance in the plate-filament circuit, as will appear below.

It is evident from what has been said that if the grid potential is varied in any manner whatever, the plate current will also vary in

proportion to, and in phase with, the grid potential variations. Thus for any instantaneous value of grid potential

$$e_g = -E_g + E_{ga} \sin \omega t, \tag{1}$$

the plate instantaneous current will be

$$i_p = I_p + I_{pa} \sin \omega t. \tag{2}$$

This shows that the alternating component of the plate-filament current is in phase with the alternating electromotive force superimposed upon the grid potential $-E_{ga}$.

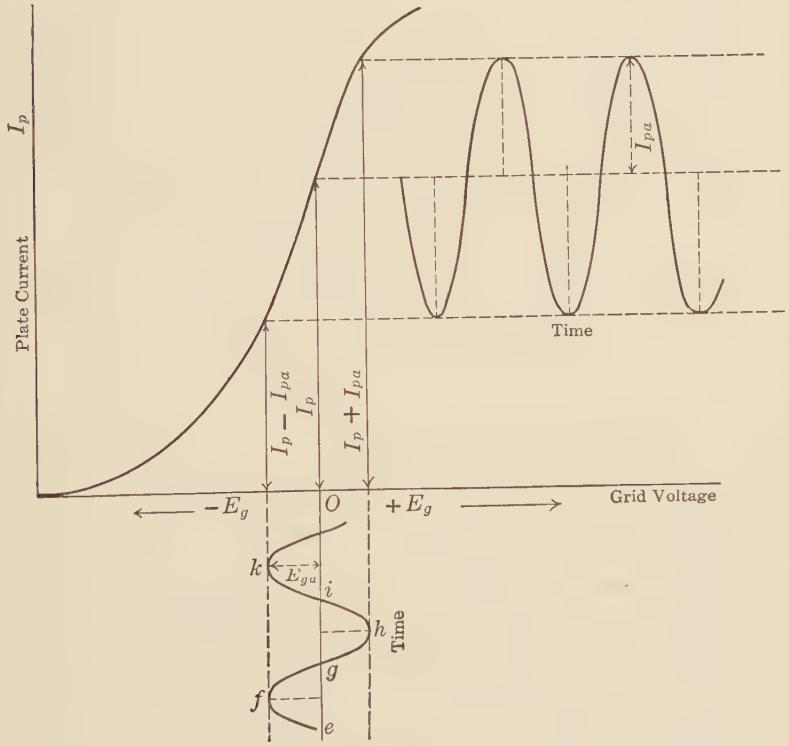

FIG. 76

The instantaneous plate-filament potential will be the plate battery voltage E_b less the voltage drop in the plate-filament circuit

$$e_p = E_b - i_p (Z_p + Z_r). \tag{3}$$

The ratio of the instantaneous plate-filament potential e_p to the instantaneous grid-filament potential e_g is called *amplification factor*, and is denoted by the symbol μ. We have, therefore

$$\mu = \frac{e_p}{e_g} = \frac{E_b - i_p (Z_p + Z_r)}{- E_g + E_{ga} \sin \omega t}. \tag{4}$$

Substituting the value of i_p from (2), we get

$$\mu = \frac{E_b - (I_p + I_{pa} \sin \omega t)(Z_p + Z_r)}{- E_g + E_{ga} \sin \omega t}, \tag{5}$$

from which it follows that

$$E_b - I_p (Z_p + Z_r) - I_{pa} (Z_p + Z_r) \sin \omega t = - \mu E_g + \mu E_{ga} \sin \omega t. \tag{6}$$

But when the superimposed alternating electromotive force on the grid potential is equal to zero ($E_{ga} \sin \omega t = 0$), the alternating component of the plate current is also equal to zero ($I_{pa} \sin \omega t = 0$). Hence, for this particular condition, equation (6) becomes

$$E_b - I_p (Z_p + Z_r) = - \mu E_g,$$

or

$$\mu = \frac{E_b - I_p (Z_p + Z_r)}{- E_g} = \frac{E_p}{- E_g}. \tag{7}$$

As the negative grid potential increases with negative superimposed electromotive force, the plate current decreases, and the potential drop $I_p (Z_p + Z_r)$ decreases also. If the battery voltage E_b is constant, this results in a higher plate potential E_p, with consequent higher electron absorbing ability. Hence the plate current does not decrease as much as it would if the voltage drop were neglected. Similarly, when the negative grid potential decreases with positive superimposed electromotive force, the plate current increases. The voltage drop $I_p (Z_p + Z_r)$ also increases, resulting in a smaller plate-filament potential. Hence the plate current does not increase so much as it would if the voltage drop were neglected. These changes in the plate-filament potential with voltage drop result in proportional variations in plate current with grid potential changes, and consequently in distortionless amplification, when the changes range over the straight portion of the curve which relates plate current to grid voltage (Fig. 74).

By definition, the internal impedance and the mutual impedance of the tube, respectively, are:

$$Z_p = \frac{E_p}{I_p}, \tag{8}$$

and

$$Z_m = \frac{E_g}{I_p}. \tag{9}$$

Hence (7) may be written in the form

$$-\mu = \frac{E_p}{E_g} = \frac{I_p Z_p}{I_p Z_m} = \frac{Z_p}{Z_m}. \tag{10}$$

This means that the amplification factor may be defined as the ratio of the internal impedance to the mutual impedance of the tube.

Since these two quantities depend upon the distance between the filament and the plate, the position of the grid with respect to both filament and plate, the size of the grid wire, and the size of the grid mesh, it is clear that the amplification factor will depend upon the structural design of the tube.

The relation between the alternating component of the plate potential and the alternating electromotive force impressed upon the grid circuit is given by the equation (6)

$$-I_{pa}(Z_p + Z_r)\sin \omega t = \mu E_{ga} \sin \omega t, \tag{11}$$

or

$$-E_{pa} \sin \omega t = \mu E_{ga} \sin \omega t,$$

which indicates that the alternating component of the plate potential is opposite in phase to the impressed alternating electromotive force upon the grid circuit. Since the alternating component of the plate current is in phase with this grid alternating electromotive force, it follows that the alternating component of the plate current is opposite in phase to the alternating component of the plate potential, and as a consequence the tube has a high power factor when considered as a translator of energy.

The instantaneous value of the alternating component of the plate current, by (11), is

$$I_{pa} \sin \omega t = \frac{\mu E_{ga} \sin \omega t}{Z_p + Z_r},$$

whence the relationship between the maximum values is

$$I_{pa} = \frac{\mu E_{ga}}{Z_p + Z_r}, \tag{12}$$

or

$$I_{pa}Z_p + I_{pa}Z_r = \mu E_{ga} \tag{13}$$

in which $I_{pa}Z_p$ is the maximum internal voltage drop due to the alternating component of the plate current, and

$$I_{pa}Z_r = E_r \tag{14}$$

is the amplitude of the useful *amplified voltage* across the receiver impedance Z_r. Substituting in equation (14) the value of I_{pa} from (12), we get

$$E_r = \mu E_{ga} \frac{Z_r}{Z_p + Z_r}. \tag{15}$$

If the useful *voltage amplification factor* is denoted by the symbol μ_1, then

$$\mu_1 = \frac{E_r}{E_{ga}} = \mu \frac{Z_r}{Z_p + Z_r}. \tag{16}$$

129. Power Amplification. — If we use the notation

$E_{rf} = E_r/\sqrt{2}$, the effective voltage across the receiver impedance, and

$I_{paf} = I_{pa}/\sqrt{2}$, the effective value of the alternating component of the plate current,

then the voltamperes output of the tube is

$$(VA)_r = E_{rf}I_{paf} = \frac{E_r I_{pa}}{2}. \tag{17}$$

Substituting in this expression the values of E_r and I_{pa}, from (15) and (12), respectively, we may write the output voltamperes in the form

$$(VA)_r = \mu^2 E^2_{ga} \frac{Z_r}{2(Z_p + Z_r)^2}. \tag{18}$$

Using the effective value of the alternating superimposed grid potential

$$E_{gaf} = \frac{E_{ga}}{\sqrt{2}},$$

equation (18) becomes

$$(VA)_r = \mu^2 E^2_{gaf} \frac{Z_r}{(Z_p + Z_r)^2}. \tag{19}$$

130. Conditions for Maximum Power Amplification. — With a fixed value of amplification factor, and constant value of effective electromotive force superimposed upon the negative grid potential, the maximum power output will depend upon the value of receiver impedance Z_r. The value of Z_r that will make the amplified receiver voltamperes a maximum may readily be obtained by differentiating equation (19) with respect to Z_r, equating the derivative to zero, and solving for Z_r. This gives

$$\frac{d\,(VA)_r}{dZ_r} = \mu^2 E^2_{gaf} \frac{(Z_p + Z_r)^2 - 2\,(Z_p + Z_r)\,Z_r}{(Z_p + Z_r)^4} = 0,$$

from which we find

$$Z_r = Z_p. \tag{20}$$

This indicates that the maximum amplified power output is obtained when the load impedance is equal to the internal impedance of the amplifier.

By substituting Z_p for Z_r in equation (19), the maximum amplified power obtainable is seen to be

$$(VA)_{mr} = \frac{\mu^2 E^2_{gaf}}{4\,Z_p}. \tag{21}$$

131. The Standard Amplifying Tube. — The filament of the three-electrode tube designed for use as an amplifier in connection with telephone lines is heated to a red heat with a normal current of 1.25 amperes, obtained from a battery connected in series with a controlling rheostat across the filament terminals. The working temperature of the filament is such that the life of the filament is about 2000 hours. The normal negative grid potential is − 9 volts, and the plate-filament battery voltage is 130 volts. Under these conditions, the plate-filament current is about 8 milliamperes and the amplification factor is about 6.

132. The Thermionic Tube as an Oscillator. — If the plate-filament output circuit of a three-electrode thermionic tube is coupled to the grid-filament input circuit, all or part of the amplified energy may be made to circulate through the thus interconnected plate and grid circuits with the tube as an intermediary. A condition analogous to that of resonance is thereby established, as may be gathered from an analysis of the interaction taking place in the oscillatory circuit shown in Fig. 77.

It will be noted that the output circuit consists of three branches in parallel:

(1) Branch a–b, including the plate battery E_b, in series with a high inductance coil L_1, the function of which is to limit the current in this branch to the direct-current component of the plate-filament current;

(2) Branch c–d, consisting of a condenser C;

(3) Branch e–f, consisting of the primary L_2 of a repeating coil, whose secondary L_3 is connected to the grid-filament circuit as shown in the figure.

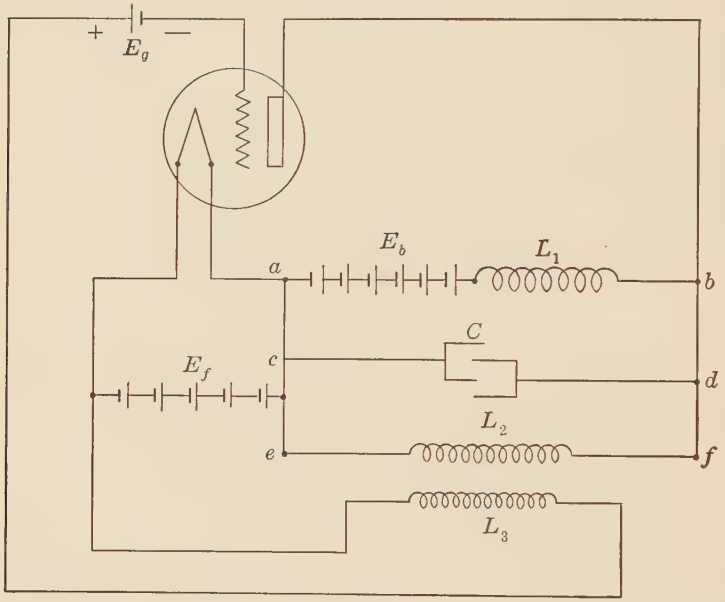

Fig. 77

If an alternating electromotive force of small amplitude is superimposed upon the constant negative grid-filament potential, an alternating-current component will result in the plate circuit, which induces an electromotive force in the repeating coil secondary L_3. This electromotive force, by virtue of the connection of L_3 with the input terminals of the tube, is superimposed upon the grid-filament negative potential, resulting in a further increase in the alternating-current component of the plate-filament circuit. In this fashion, the output plate-filament circuit will react with the input

grid-filament circuit, until a maximum amplification has been reached, and the system settles down to a steady oscillatory state.

From the nature of the electrical properties of the branches c–d and e–f of the plate circuit, the currents in the two branches, under steady conditions of oscillation, will be 180° opposite in phase to each other, since the alternating component of the plate-filament potential is the same across each branch and we have assumed the resistance of each to be negligible.

At the instant the alternating voltage across c–d and e–f is zero, the current through the condenser is a maximum. The condenser is just beginning to charge. The current through the inductance L_2 is also a maximum, but opposite in phase to that in branch c–d, all the energy of the branches c-d and e-f being stored in the magnetic field of L_2. As the instantaneous potential across c–d and e–f increases on account of the transfer of energy from the magnetic field of L_2 to the electrostatic field of C, the current through L_2 decreases, and the magnetic field decreases with it. The rate of

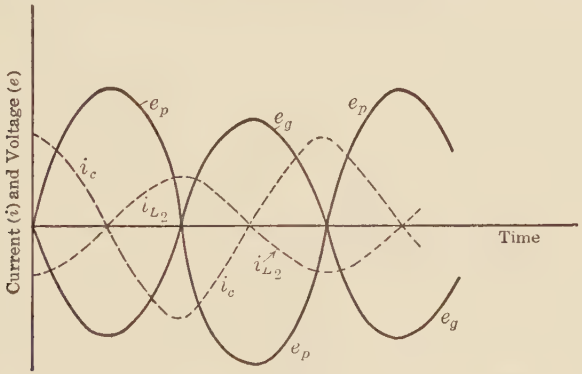

FIG. 78

change of interlinkage of this field with the turns of the secondary L_3 results in a generated electromotive force in L_3, 180° out of phase with the potential across L_2. This electromotive force is superimposed upon the constant negative grid-filament potential. Part of the energy stored electromagnetically in L_2 is thus delivered through L_3 to the grid and thence through the agency of the tube to the condenser. At the instant the current in L_2 is zero, the rate of change of interlinkage of the field of L_2 with the turns of L_3 is a maximum; the grid potential is a maximum; the plate potential

is also a maximum, but is opposite in phase, and the condenser is fully charged, the current through it being zero. As the grid-filament potential decreases, the plate alternating potential, still opposite in phase, also decreases; the condenser discharges into the inductance L_2; the currents in the two branches are at maximum values, opposite in phase to one another, and to their previous phases, respectively.

The relationship between the instantaneous currents and potentials in the branches of the circuit is shown in Fig. 78.

It is seen from what precedes that the thermionic tube and the associated circuit will produce a circulation of energy with a periodic conversion of it from the electrostatic state in the condenser branch to the electromagnetic state in the repeating coil L_2L_3, and vice versa. The important result of this arrangement is that in the combination of the branches c–d and e–f we have *self-sustained electric oscillations*, with the energy losses due to the flow of current through the resistances of the circuits, supplied by the current from battery E_b as it flows through the tube.

Under the assumption of negligible resistance, the admittance of the two branches is

$$Y_{e-f} = \sqrt{\frac{C}{L_2}}, \tag{22}$$

which is the natural admittance of the circuit. If the inductive susceptance of branch e–f is made equal to the dielectric susceptance of branch c–d, we have

$$\frac{1}{L_2\omega} = C\omega,$$

or

$$\omega = \frac{1}{\sqrt{L_2C}},$$

whence the frequency of the self-sustained oscillations is

$$f = \frac{1}{2\pi\sqrt{L_2C}}, \tag{23}$$

which is the natural frequency of the circuit.

133. The Oscillator on Load. Operating Conditions of the Thermionic Oscillator. — The oscillator described in the previous section may be thought of as an alternator on no load. Some

energy is required to establish the exciting static field, but no energy is required to maintain it beyond that dissipated in the ohmic resistance of the circuit.

If we consider that the energy is actually derived from the plate battery E_b, the device may be thought of, rather, as a *converter*, since it converts a steady unidirectional potential into an alternating potential.

To load the oscillator, a resistance or impedance load may be inserted in various ways, as for instance in one of the branches *c–d* or *e–f* or in the grid circuit or between the terminals of L_3. The conditions to be imposed for the occurrence of self-sustained oscillations under load conditions may be determined from the relation between the quantities involved.

Thus, consider a resistance load R inserted in branch *e–f* in series with L_2, Fig. 77. Let us set

E_b = constant plate-filament potential, due to battery in branch *a–b* producing a constant electrostatic field between filament and plate;

$-E_g$ = constant negative grid-filament potential, producing a constant static field opposed to and superimposed upon the static field produced by E_b;

I_p = plate-filament current resulting from the constant excitation due E_b and $-E_g$, and which is confined to branch $a-b$;

E_{ga} = effective generated electromotive force in secondary L_3, the result of the alternating current in L_2, and which superimposed upon $-E_g$, produces an alternating plate-filament electrostatic field superimposed upon the constant field due to E_b and $-E_g$ combined;

I_{ap} = the effective value of the generated plate-filament alternating current due to E_{ga}, and which is superimposed upon I_p. This alternating component of the plate-filament current flows against the internal resistance (impedance) of the tube in series with the combined impedance of branches *c–d* and *e–f*;

I_1 = the component of I_{ap} through condenser branch *c–d*;

I_2 = the component of I_{ap} through the inductive branch *e–f*;

E_p = effective value of voltage in the plate-filament circuit due to I_{ap};

E_{12} = component of E_p across branches *c–d* and *e–f*;

R_p = internal resistance of the tube*;
μ = amplification factor of the tube;
N_2 = turns of primary inductance, L_2;
N_3 = turns of secondary inductance, L_3;
n = N_3/N_2 = ratio of transformation of repeating coil;
R = load resistance, in series with L_2.

Considering the effective values of the alternating voltages and currents, conventionally, as vectors, and the operations to be carried out as vector operations, the following relations may be established between the quantities involved:

$$E_{12} = E_p - I_{ap}R_p, \tag{24}$$

and since we have by (7), $E_p = -\mu E_{ga}$, (24) may be written in the form

$$E_{12} = -\mu E_{ga} - I_{ap}\,R_p. \tag{25}$$

Since E_{ga} is the electromotive force generated in the secondary L_3, its value is $n \times$ voltage across L_2, and is 180° out of phase with the voltage across L_2. But the voltage across L_2 is leading the current I_2 by 90°, and its value is

$$\text{voltage across } L_2 = jL_2\omega I_2;$$

hence

$$E_{ga} = -jnL_2\omega I_2. \tag{26}$$

Substituting in (25), we find

$$E_{12} = j\mu nL_2\omega I_2 - I_{ap}R_p. \tag{27}$$

Since the alternating-current component in the plate-filament circuit is the vector sum of the currents in the branches *c–d* and *e–f*, and since also

$$I_1 = +jC\omega E_{12}, \tag{28}$$

and

$$I_2 = \frac{E_{12}}{R + jL_2\omega}, \tag{29}$$

it follows that (27) may be written in the form

$$E_{12} = \frac{j\mu nL_2\omega}{R + jL_2\omega}\,E_{12} - jC\omega R_p E_{12} - \frac{R_p}{R + jL_2\omega}\,E_{12},$$

* The internal impedance of a tube is usually not appreciably different from the internal resistance, and the succeeding formulas, therefore, are written with R_p only. In case of appreciable internal reactance, Z_p must be substituted for R_p in the vector equations.

which reduces to

$$\frac{j\mu nL_2\omega - R_p}{R + jL_2\omega} = 1 + jC\omega R_p. \tag{30}$$

Multiplying out, we get

$$j\mu nL_2\omega - R_p = R + jL_2\omega + jCRR_p\omega - \omega^2 L_2 CR_p. \tag{31}$$

Equating reals and imaginaries, respectively, on both sides of the equality sign, we obtain

$$\omega^2 L_2 CR_p = R + R_p, \tag{32}$$

$$\mu nL_2 = L_2 + CRR_p. \tag{33}$$

An analysis of these two fundamental equations of the thermionic oscillator will indicate the conditions under which self-sustained oscillations can be established and maintained indefinitely. Thus, solving equation (32) for ω, we get

$$\omega = \sqrt{\frac{R + R_p}{L_2 CR_p}},$$

whence the frequency of the oscillations is

$$f = \frac{1}{2\pi}\sqrt{\frac{1 + \frac{R}{R_p}}{L_2 C}}. \tag{34}$$

This equation indicates that if the load resistance R is sensibly equal to zero, the frequency approaches the natural frequency of the circuit. This checks with the results of the analysis of the oscillator on no load given in the preceding section. It assumes that the internal reactance of the tube is negligible.

Equation (34) shows also that with a fixed value of L_2C, the frequency of the oscillations increases with the load resistance. It becomes about 41 per cent larger than the natural frequency when the load resistance is equal to the internal tube resistance, for under that condition

$$f = \frac{1}{2\pi}\sqrt{\frac{2}{L_2 C}}.$$

It will be observed also that with a fixed load resistance, the frequency of the oscillations may be varied by changing the value of L_2C, either by varying the value of L_2 or of C or of both. In this manner the frequency may be varied from a fraction of a cycle to several millions of cycles per second.

Solving equation (33) for the ratio of transformation of the repeating coil, we get

$$n = \frac{1}{\mu} + \frac{CR_p}{\mu L_2} R. \tag{35}$$

With the load resistance R equal to 0, this equation becomes

$$n = \frac{1}{\mu}. \tag{36}$$

This shows that, no matter what the values of the fixed positive plate potential and the constant negative grid potential may be, the ratio of transformation of the repeating coil must not be less than the reciprocal of the tube amplification factor. In other words, the number of repeating-coil primary turns must not be less than the number of secondary turns multiplied by the amplification factor of the thermionic oscillator

$$N_2 = \mu N_3. \tag{37}$$

The value of n given by equation (36) may then be considered as a critical value, for if it is smaller than the reciprocal of the amplification factor, no self-sustained oscillations can be set up, even with the oscillator on no load. With larger values of n than that given by equation (36), steady oscillations will be set up, with limiting amplitudes depending, for a given tube and plate battery voltage, on the values of grid constant negative potential $-E_g$ and the constants of the circuit.

Equation (35) gives the critical value of the ratio of transformation for any value of load resistance R. As the load resistance is increased, the ratio of transformation must be increased above the critical value of n for the no-load condition, by an amount directly proportional to the increase in load resistance, the constant of proportionality being $CR_p/\mu L_2$. With any definite value of load resistance R, no oscillations will be set up, if n is smaller than that given by equation (35). With larger values of n, the steady oscillations will increase in amplitude to the limiting value, which is determined, for a given tube and plate-battery voltage, by the value of $-E_g$ and the constants of the circuit.

CHAPTER XV

REPEATER OR AMPLIFYING CIRCUITS

The electric circuit comprising one or two amplifiers and their batteries, rheostats, condensers, induction coils, through which amplifiers are connected to a telephone line, and the balancing artificial network when one is used, is called an **amplifying circuit.**

134. One-way Amplification. — The simplest amplifying circuit is a circuit associated with a line for **one-way** operation. Thus, consider a long aerial line closed through an impedance equal to the characteristic line impedance. Since such a line is equivalent in its behavior to an infinitely long line, there are no terminal reflections, and the energy attenuates with the uniformity characteristic of a line of this type.

If the energy is to be amplified from some local source at some point S miles from the sending-end A, Fig. 79, the line is severed at that point and the amplifier is connected in, with its input terminals to the incoming line from A and the output terminals to the outgoing line to B. Since the amplifier output depends upon the voltage superimposed upon the negative grid-filament potential, the voltamperes input must be changed to a comparatively high-potential input. This is accomplished by connecting the amplifier input terminals to the incoming line through a repeating coil T_1, Fig. 79, with the high-potential secondary on the amplifier side, generally shunted by a high resistance R_1, numerically equal to the secondary impedance of the repeating coil. For reasons given in § 121, Chapter XIII, the primary of this repeating coil must be equal to the line impedance which for the particular line under consideration is the characteristic impedance.

Since only the alternating component is the useful current output of the amplifier, it is necessary that the direct-current output be filtered out. This is accomplished by means of the output repeating coil T_2, whose primary, for maximum amplifier output, must be equal to the internal impedance of the amplifier, and whose secondary must be equal to the line impedance, so that the uniform

line properties remain unimpaired. The direct-current component of the amplifier current output may be eliminated also from the primary of T_2, and thus permit its operation at a lower temperature. This is accomplished by the use of a filter made up of a condenser C_2 in series with the primary of T_2, and the two so connected shunted by a high-inductance coil L_2.

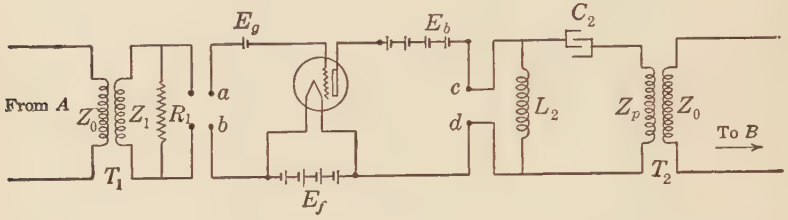

FIG. 79

To find the amplification obtainable by means of this amplifying circuit, let us set

a = the attenuation constant of the line;

E_s = effective sending-end voltage at terminus A of the line;

I_s = effective sending-end current at that terminus.

The voltamperes supplied to the primary of the input repeating-coil T_1 is

$$(VA)_1 = E_s I_s \epsilon^{-2aS}. \tag{1}$$

If n is the ratio of transformation of the repeating coil T_1, then with negligible leakage reactance in this repeating coil, the secondary voltage, which is also the voltage superimposed upon the negative grid-filament potential, is

$$E_{ga} = n E_s \epsilon^{-aS}. \tag{2}$$

Hence, by (21), Chapter XIV, the amplified voltamperes output is

$$(VA)_2 = \frac{\mu^2 E_{gaf}^2}{4 Z_p} = \frac{\mu^2 n^2 E_s^2 \epsilon^{-2aS}}{4 Z_p}. \tag{3}$$

Assuming negligible losses in the repeating coil T_2, it is obvious that (3) represents also the amplified voltamperes output into the line towards the B terminus.

The *power amplification*, which is the ratio of the voltamperes output to the voltamperes input, is therefore

$$\text{power amplification} = \frac{\mu^2 n^2 E_s^2 \epsilon^{-2aS}}{4 Z_p E_s I_s \epsilon^{-2aS}} = \frac{\mu^2 n^2 E_s}{4 Z_p I_s}.$$

Since $Z_s = Z_0$ for this particular line, it follows that

$$\text{(4)} \qquad \text{power amplification} = \frac{\mu^2 n^2}{4} \cdot \frac{Z_0}{Z_p}.$$

Since $n^2 = Z_1/Z_0$ (see § 121, Chapter XIII), where Z_1 is the impedance of secondary of repeating coil T_1, it follows by substituting in (4), that

$$\text{(5)} \qquad \text{power amplification} = \frac{\mu^2}{4} \cdot \frac{Z_1}{Z_p}.$$

In an amplifying circuit in which $Z_1 = 600{,}000$ ohms, $Z_p = 30{,}000$ ohms, and $\mu = 7$, the power amplification would be

$$\frac{7^2}{4} \times \frac{600{,}000}{30{,}000} = 245;$$

in other words, the output would be 245 times the input.

Since both voltage and current attenuate at the same rate for the type of line under consideration ($Z_r = Z_0$), they will also amplify equally, the amplification for each being

$$\text{voltage (or current) amplification} = \frac{\mu}{2} \sqrt{\frac{Z_1}{Z_p}}.$$

For the case just discussed, this value will be 15.66

If an amplifying circuit that has the constants assumed above were connected in at a point 226.4 miles from the sending end of our illustrative No. 10 B. & S. line, the voltage, current, and volt-amperes before amplifying would be

$$\begin{aligned} E_s\epsilon^{-2aS} &= \quad 50 \times .1275 = 6.375 \text{ volts}, \\ I_s\epsilon^{-2aS} &= .0692 \times .1275 = .00882 \text{ amperes}, \\ (VA)_1 &= 6.375 \times .00882 = .0562 \text{ voltamperes}, \end{aligned}$$

whence the amplified values would be

$$\begin{aligned} \text{the amplified voltage} &= 15.66 \times 6.375 = 99.78 \text{ volts}, \\ \text{the amplified current} &= 15.66 \times .00882 = .1381 \text{ amperes}, \\ \text{amplified voltamperes} &= 245 \times .0562 = 13.77 \text{ voltamperes}. \end{aligned}$$

135. Two-Way, One-Element, Amplifying Circuit. — The amplifying circuit discussed in the preceding section, being unidirectional in its action, is not suitable for use in connection with telephone lines. The terminals of a telephone line are intermittently sending and receiving ends, depending on the conversation carried on over

it. An amplifying circuit, to operate successfully in connection with it, must amplify intermittently in both directions. One possible scheme would be the use of two separate and independent lines between termini, each with its own one-way amplifying circuit. Such a system, however, is practically out of the question except in long line cable circuits, as it would mean doubling the number of circuits between termini. Figure 80 shows diagrammatically how a single amplifying circuit can be used in connection with a telephone line to bring about amplification in both directions.

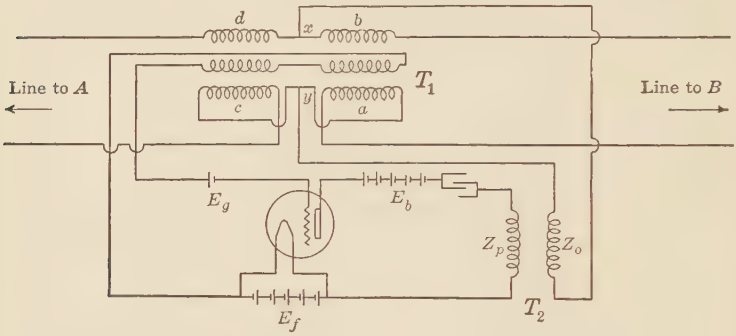

Fig. 80

It will be observed, from an examination of the above diagram, that the primary of the input repeating coil T_1 consists of four coils. Two of these coils, b and d, are connected in series with one line wire, and the other two, a and c, are connected in series with each other, and with the other line wire. The last two coils are differentially wound with respect to coils b and d, respectively, so that the resultant magnetomotive force is the sum of the magnetomotive forces produced by each coil. The secondary of this repeating coil is connected to the input terminals of the amplifier tube, while the primary of the repeating coil T_2 is connected to the output terminal of the amplifier tube. The secondary of repeating coil T_2 is connected across the line at xy, the junctions of coils a–c and b–d, respectively. From the nature of the connections, it is clear that the input circuit of the amplifier is in series with the line through the repeating coil T_1, and the output circuit of the amplifier is connected in parallel with the line through the repeating coil T_2. Hence the scheme may be called *series-parallel amplification.*

To find the resulting amplification with this particular amplification circuit, let us set

I_s = sending-end current at the A terminus of the line;
a = the line attenuation constant;
S = the distance from terminal A to the amplifier station.

Under the assumption that the receiver impedance at the terminals are equal to the characteristic line impedance, so that the energy will attenuate hyperbolically along the characteristic curve pertaining to the infinite line, the current at the amplifier station in the primary of the input repeating coil T_1 will be $I_s\epsilon^{-aS}$.

If the ratio of transformation of the step-up repeating coil T_1 is n_1, then the current in the secondary of this repeating coil is $I_s\epsilon^{-aS}/n_1$. If Z_2 is the impedance of secondary of the repeating coil T_1, the voltage across this secondary, which is also the voltage superimposed on the grid-filament negative battery potential, is

$$E_{ga} = \frac{I_s Z_2 \epsilon^{-aS}}{n_1}. \tag{6}$$

Assuming that the primary of the step-down repeating coil T_2 has an impedance equal to the internal impedance of the amplifier tube, we see by (21), Chapter XIV, that the maximum amplified voltamperes will be

$$(VA)_m = \frac{\mu^2}{4}\frac{E_{ga}^2}{Z_p}.$$

Substituting the value of E_{ga} from (6), we get

$$(VA)_m = \mu^2\frac{I_s^2 Z_2^2 \epsilon^{-2aS}}{4\, n_1^2 Z_p}. \tag{7}$$

Assuming negligible losses in the output repeating coil T_2, it is evident that (7) represents also the maximum amplified voltamperes supplied to the line at xy.

If $n_1^2 = Z_2/Z_0$ (see § 121, Chapter XIII), substituting the value of Z_2 from this equation in (7), we get

$$(VA)_m = \frac{\mu^2}{4}\frac{Z_0^2}{Z_p} n_1^2 I_s^2 \epsilon^{-2aS}. \tag{8}$$

By (15), Chapter XIV, the amplified voltage across the primary of T_2 is

$$E_3 = \frac{\mu}{2} E_{ga} = \frac{\mu}{2}\frac{I_s Z_2 \epsilon^{-aS}}{n_1} = \frac{\mu}{2} n_1 I_s Z_0 \epsilon^{-aS}. \tag{9}$$

If the ratio of transformation of step-down repeating coil T_2 is n_2, then the secondary voltage, which is also the voltage across the line at xy, is

$$E_4 = \frac{E_3}{n_2} = \frac{\mu}{2}\frac{n_1}{n_2} I_s Z_0 \epsilon^{-aS}. \tag{10}$$

Hence, by (8) and (10), the total current supplied to the line at xy will be

$$I_4 = \frac{(VA)_m}{E_4} = \frac{\mu}{2}\frac{Z_0}{Z_p} n_1 n_2 I_s \epsilon^{-aS}. \tag{11}$$

Since the line to A is in parallel with the line to B with respect to the output current of the amplifier circuit, it follows that the current given by (11) will divide inversely as the impedances of the two lines as measured at xy, and will flow in opposite directions in the coils d and b and also c and a, respectively.

If then the impedance of the line to A and that of the line to B as measured at xy are not equal, there will be an unbalanced magnetomotive force in the primary of T_1, resulting in an induced electromotive force in its secondary, which is superimposed on the grid-filament potential. The amplifier will operate as an oscillator, and will cause distortion of the amplified waves.

The successful operation of this type of amplifying circuit calls for a perfect equality between the currents toward the opposite terminals, *i.e.*, a perfect balance between the impedances of the lines to A and B as measured at xy. With a uniform homogeneous line, this can be accomplished by locating the amplifying circuit just midway between the line termini.

It will be evident from what has been said, that only half of the total amplified voltamperes, namely, that towards the receiving end, is useful, the other half being dissipated in the line towards the sending end. The maximum useful amplified voltamperes will therefore be

$$(VA)_{mu} = \frac{(VA)_m}{2} = \frac{\mu^2 Z_0^2}{8 Z_p} n_1^2 I_s^2 \epsilon^{-2aS}. \tag{12}$$

The useful amplification is the ratio of the useful voltamperes output to the voltamperes input

$$\text{useful amplification} = \frac{(VA)_{mu}}{E_s I_s \epsilon^{-2aS}} = \frac{\mu^2 Z_0^2}{8 Z_p} \frac{n_1^2 I_s^2 \epsilon^{-2aS}}{E_s I_s \epsilon^{-2aS}} = \frac{\mu^2}{8}\frac{Z_0^2}{Z_p}\frac{n_1^2 I_s}{E_s}.$$

Since we have, for a line with the characteristics of the infinite line,

$$\frac{I_s}{E_s} = \frac{1}{Z_0},$$

it follows that the useful amplification is

$$\text{useful amplification} = \frac{\mu^2}{8}\frac{Z_0}{Z_p}n_1^2. \tag{13}$$

136. Two-way, One-Element, Parallel-Series Amplification. — Another and more common scheme of connecting an amplifier circuit to a telephone line is what may be called *parallel-series amplifying circuit*, which is shown diagrammatically in Fig. 81.

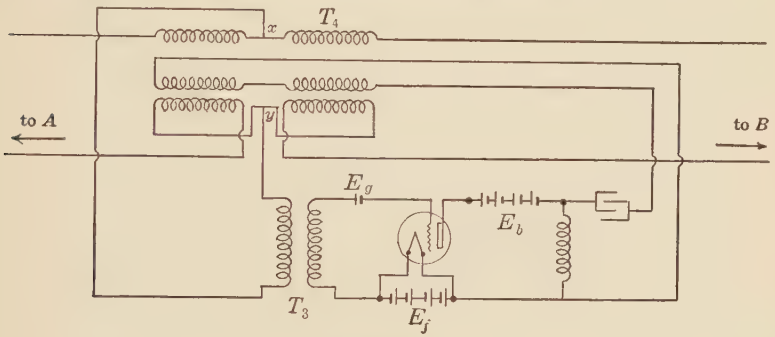

FIG. 81

It will be observed from this diagram that the input circuit of the amplifier is connected in parallel with the line at xy through a step-up repeating coil T_3, and that the amplifier output circuit is in series with the line through the step-down repeating coil T_4.

If E_s is the sending-end voltage at the A terminus of the line, then, assuming that the line is uniform, having the characteristics of the infinitely long line, and that the amplifier station is located at a distance S from the A terminus, the voltage at xy will be $E_s\,\epsilon^{-aS}$. If n_3 is the ratio of transformation of repeating coil T_3, the voltage superimposed on the negative grid-filament potential will be

$$E_{ga} = n_3 E_s \epsilon^{-aS}. \tag{14}$$

With the primary impedance of T_4 equal to the internal impedance of the amplifier tube, the maximum amplified voltamperes will be

$$(VA)_m = \frac{\mu^2 E_{ga}^2}{4\,Z_p} = \mu^2 \frac{n_3^2 E_s^2\,\epsilon^{-2aS}}{4\,Z_p}. \tag{15}$$

Assuming negligible losses in the repeating coil T_4, this will also represent the voltamperes supplied to the line.

The amplified voltage in the primary of T_4 is

$$E_4 = \frac{\mu}{2} E_{ga} = \frac{\mu}{2} n_3 E_s \epsilon^{-aS}, \tag{16}$$

whence the current in the primary of T_4 will be

$$I_4 = \frac{(VA)_m}{E_4} = \frac{\mu}{2} \frac{n_3 E_s \epsilon^{-aS}}{Z_p}. \tag{17}$$

If n_4 is the ratio of transformation of step-down repeating coil T_4, the current in its secondary will be

$$I_5 = n_4 I_4 = \mu n_3 n_4 \frac{E_s \epsilon^{-aS}}{2}, \tag{18}$$

and this is the total amplified current supplied to the line.

If Z_A and Z_B are the impedances of the line to A and B, respectively, as measured at xy, the potential across the line due to I_5 towards A is $I_4 Z_A$, and the potential of the line to B is $I_4 Z_B$. Hence, to prevent the amplifier from acting as an oscillator and thereby causing distortion, the potential at xy due to this amplified current must be equal to zero, *i.e.*, $I_4 Z_A = I_4 Z_B$. It follows therefore that Z_A must be equal to Z_B. With a uniform homogeneous line this condition can be obtained by locating the amplifier station midway between the termini.

As with the series-parallel amplifying circuit, only one-half of the amplified energy is useful, namely, that towards the receiving end, the other half being dissipated in the line towards the sending end. Hence, by (15), the maximum useful amplified voltamperes is

$$(VA)_{mu} = \mu^2 \frac{n_3^2 E_s^2 \epsilon^{-2aS}}{8 Z_p}, \tag{19}$$

and the useful amplification is

$$\frac{(VA)_{mu}}{E_s I_s \epsilon^{-2aS}} = \mu^2 \frac{n_3^2 E_s^2 \epsilon^{-2aS}}{8 Z_p E_s I_s \epsilon^{-2aS}} = \mu^2 \frac{n_3^2 E_s}{8 Z_p I_s}.$$

Since we have, for a line with the characteristics of the infinite line, $E_s/I_s = Z_0$, it follows that the useful amplification is

$$\text{useful amplification} = \mu^2 \frac{n_3^2 Z_0}{8 Z_p}. \tag{20}$$

137. Two-way, Two-Element, Amplifying Circuit. — The two-way, one-element, amplifying circuit discussed in the preceding two sections is commonly called type "21." It is obvious that its use must be confined to lines of such length that a single amplifier is sufficient. It is not suitable for parallel operation, on account of the distortioning circulating currents set up between adjacent amplifiers which act partially as oscillators.

A two-way, two-element, amplifying circuit, universally used in connection with lines requiring more than one amplifying station, is shown diagrammatically in Fig. 82. As seen from the diagram, this circuit consists of two independent parallel-series amplifying circuits, each one amplifying in one direction only. Such a circuit is called a type "22" circuit.

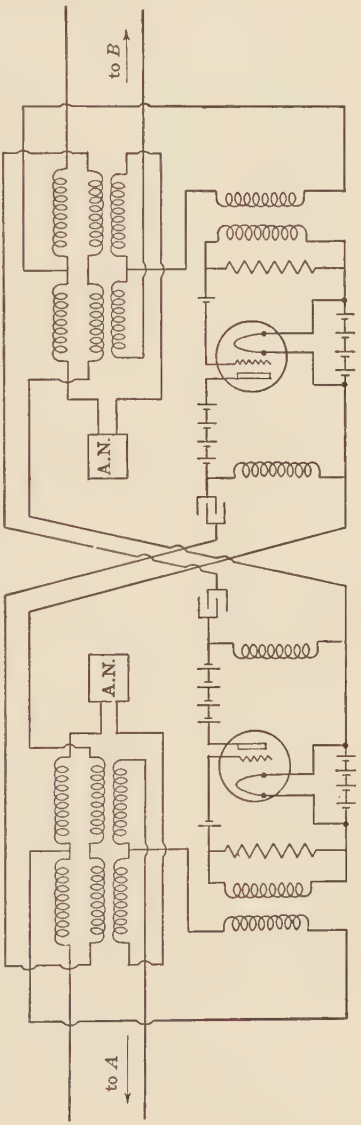

Fig. 82

With the type "21" circuit, oscillatory action of the amplifier, and the distortion produced by such action, is avoided by locating the amplifier station midway between the termini, balancing the two halves of line against each other. With type "22" circuit, however, each line must be balanced by an artificial circuit indicated by *A.N.* in Fig. 82. Such a circuit is usually called a *balancing network*. The impedance of the artificial network must be equal to the line impedance for all frequencies within the range of voice frequencies. Under these conditions, the useful

amplified voltamperes for each of the harmonics of a train of telephonic waves is one-half of the total amplified voltamperes for that harmonic, the other half being dissipated in the artificial network *A.N.*, Fig. 82.

Four-wire circuits with terminal balancing obtained as in the type "22" circuit and one-way amplifiers in tandem as required in the two transmission paths respectively are used in long distance cable circuits.

CHAPTER XVI

LINE IMPEDANCE AND ITS MEASUREMENT

138. Variation of Line Impedance with Frequency. — Since the successful operation of type "22" amplifying circuit depends largely upon a perfect balance between the artificial network and the line, for all frequencies within the range of speech frequencies, the design and construction of the network calls for a knowledge of the variation of line impedance with frequency.

Let us assume a non-loaded line closed over an impedance equal to the characteristic line impedance in modulus and in phase, thereby making the line equivalent in its behavior to an infinitely long one. The line impedance, as measured at the testing end, will be equal to the characteristic line impedance at test frequency

$$Z_0 = \sqrt{\frac{R + jL\omega}{G + jC\omega}}.$$

Let us assume also that the line conductors are of uniform cross-sectional area, equally spaced for the entire length of line of uniform insulation resistance, and that there is no interference due to neighboring telephone or power circuits. Then it is evident that the line impedance will be a function of the frequency only. The law of variation is as shown by curve 1, Fig. 59, Chapter XII. This curve shows that the line impedance decreases slightly with increase in frequency for speech frequencies lower than 600 cycles per second, and is more or less independent of the frequency for frequencies higher than the average speech frequency of 796 cycles per second.

The artificial network to balance such a line must necessarily have an impedance that would vary with the frequency along a curve similar in shape to the impedance frequency curve of the line itself.

If the line is ideally loaded, and thus perfectly distortionless, its characteristic impedance is

$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{R}{G}}.$$

The characteristic phase angle is equal to zero. The line impedance is therefore constant and independent of the frequency. Consequently, the artificial network to balance it must, obviously, be a non-inductive ohmic resistance, numerically equal to $\sqrt{R/G}$.

The variation of the characteristic impedance of the regularly series-loaded line with frequency may be studied by an analysis of the characteristic impedance of the smooth line equivalent to the loaded line in its relationship to the frequency. The characteristic impedance of the smooth line equivalent to the loaded line, by (18), Chapter XIII, is

$$(1) \quad Z_{q0} = \sqrt{Z_0^2 + Z_l^2 + 2\,Z_0 Z_l \operatorname{ctnh} p_0 S_d}$$

$$= \sqrt{Z_0^2 + Z_l^2 + 2\,Z_0 Z_l \sqrt{\frac{\cosh^2 a_0 S_d - \sin^2 v_0 S_d}{\cosh^2 a_0 S_d - \cos^2 v_0 S_d}}},$$

where the quantities involved are the loading-coil impedance and the constants of the same line before loading.

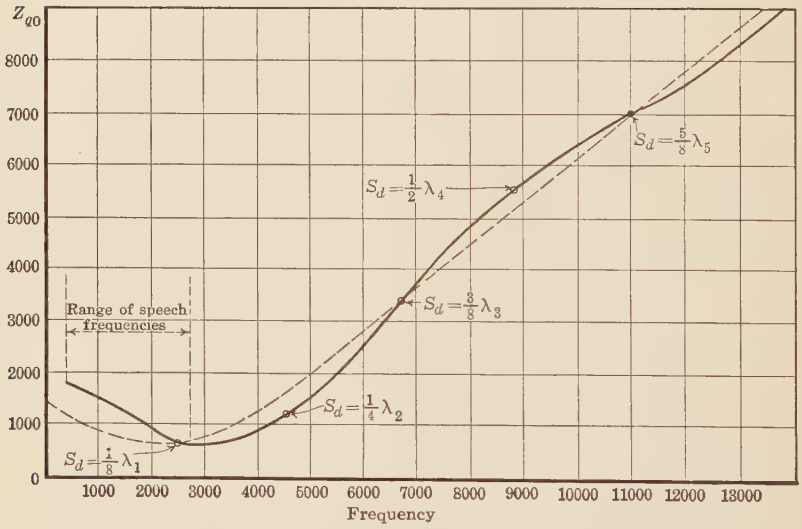

FIG. 83. Variation of Characteristic Impedance with Frequency. Regularly Series-loaded Line

This would give a smooth impedance-frequency curve, if it were not for the hyperbolic factor in the third term of the equation. This factor indicates that the line impedance passes through maximum values when the frequency is such as to make the spacing

distance between loading coils an integral number of half wave-lengths, and it will pass through minimum values for all frequencies that make the spacing distance equal to any odd number of quarter wave-lengths, as shown by the full curve, Fig. 83. On the other hand, for all frequencies that will make the spacing distance S_d equal to an odd number of eighths of a wave-length, the line impedance will be

$$Z_{q0} = Z_0 + Z_l.$$

These particular values of impedance are given by the intersections of the full and dotted curves, Fig. 83.

Considering then what has been said above, the impedance-frequency curve of a regularly series-loaded line is not a smooth curve, but will pass through maxima and minima as the line goes through half-wave and quarter-wave resonance. However, since the spacing distance S_d between the loading coils is small in comparison with the wave-lengths of the component harmonics of voice waves, the frequency intervals between maxima and minima is rather large. The portion of the impedance-frequency curve for such a line that lies within the limits of speech frequencies is practically a smooth curve. Therefore, a balancing artificial network may be constructed which will approximate very nearly the impedance-frequency curve of the regularly series-loaded line for all frequencies within the range of speech frequencies.

Perfectly uniform lines, however, do not occur in practice. Irregularities will usually occur, such as a stretch of line of different transmission constants than the main line. For instance, a cable line may be in series with an aerial line, either at its terminals or somewhere along its length. A change may occur in the interaxial distance between the wires of an open aerial line, resulting in a non-uniform distribution of inductance and capacitance. A change may occur in the leakage conductance at some point along the line. All of these possible irregularities are more or less insignificant in comparison with such irregularities as loading coils of non-uniform impedance with regard to coil resistance or more likely coil inductance, and particularly non-uniform spacing between loading coils.

With one or more such irregularities, the impedance-frequency curve of such a line will not be a smooth curve. Since the artificial network is designed and constructed to balance a uniform line, the balancing will be imperfect, and the amplifiers will not

operate successfully on account of the oscillatory action set up and the distortional effect produced thereby.

139. Effect of Irregularities on the Impedance-Frequency Curve of a Line. — To determine the change in slope of the impedance-frequency curve of a line that results from any irregularity, assume a non-loaded line with an extreme sort of an irregularity, such as an open circuit or a short-circuit at some point S miles from the sending end.

In the case of an open-circuited non-loaded line, the tensor value of the impedance at the sending end, by (3), Chapter VI, is

$$Z_s = \frac{Z_0}{\tanh pS} = Z_0 \sqrt{\frac{\cosh^2 aS - \sin^2 vS}{\cosh^2 aS - \cos^2 vS}}. \tag{2}$$

The attenuation constant a is more or less independent of the frequency, for frequencies higher than the average speech frequency of 796 cycles per second (curve a, Fig. 61). Hence, taking the velocity of propagation also as a constant for frequencies above 800 (curve V, Fig. 61), equation (2) may be written in the form

$$Z_s = Z_0 \sqrt{\frac{K - \sin^2 \frac{2\pi fS}{V}}{K - \cos^2 \frac{2\pi fS}{V}}}, \tag{3}$$

where $K = \cosh^2 aS$. An analysis of equation (3) will indicate that Z_s has as its maximum the value

$$Z_s = Z_0 \sqrt{\frac{K}{K-1}}, \tag{4}$$

when

$$\sin^2 \frac{2\pi fS}{V} = 0, \text{ and } \cos^2 \frac{2\pi fS}{V} = 1,$$

i.e., when

$$\frac{2\pi fS}{V} = n\pi,$$

or when

$$f = \frac{nV}{2S}, \tag{5}$$

where n is any integer. From (5) it is obvious that the sending-end impedance with the line open-circuited at a point S miles from the

sending end will pass through maxima for values of frequencies that make the line just equal to half a wave-length, or a multiple thereof, resulting in half-wave resonance.

Similarly, the sending-end impedance with the line open-circuited will have a minimum equal to

$$Z_s = Z_0 \sqrt{\frac{K-1}{K}} \tag{6}$$

when

$$\sin^2 \frac{2\pi fS}{V} = 1, \quad \cos^2 \frac{2\pi fS}{V} = 0,$$

i.e., when

$$\frac{2\pi fS}{V} = (2n-1)\frac{\pi}{2},$$

or when

$$f = \frac{(2n-1)V}{4S}. \tag{7}$$

This indicates that the sending-end impedance will pass through minima for values of frequency that make the open-circuited line equal to an odd number of quarter wave-lengths, resulting in quarter-wave resonance.

On the other hand, when

$$\sin^2 \frac{2\pi fS}{V} = \cos^2 \frac{2\pi fS}{V},$$

that is, when

$$\frac{2\pi fS}{V} = (2n-1)\frac{\pi}{4},$$

or when

$$f = \frac{(2n-1)V}{8S}, \tag{8}$$

the sending-end impedance will be equal to the characteristic line impedance at that frequency, *i.e.*, $Z_s = Z_0$. It is evident that this occurs when the frequency is such that the open-circuited line is equal to an eighth of a wave-length, or an odd number thereof.

Consider now the same line short-circuited at a point S miles from the sending end. The tensor value of the sending-end impedance of such a line, by (3), Chapter VII, is

$$Z_s = Z_0 \tanh pS = Z_0 \sqrt{\frac{\cosh^2 aS - \cos^2 vS}{\cosh^2 aS - \sin^2 vS}},$$

Following the same method of analysis as that given for the open-circuited line, we see that the sending-end impedance of a line having such an extreme sort of an irregularity as a short circuit, will also pass through maxima and minima, the maxima occurring when

$$\frac{2\,\pi f S}{V} = (2\,n - 1)\,\frac{\pi}{2},$$

i.e., when

$$f = \frac{(2\,n - 1)\,V}{4\,S}, \tag{9}$$

and the minima when

$$\frac{2\,\pi f S}{V} = n\pi,$$

i.e., when

$$f = \frac{n V}{2\,S}. \tag{10}$$

On the other hand, when

$$f = \frac{(2\,n - 1)\,V}{8\,S}, \tag{11}$$

the sending-end impedance will be equal to the characteristic line impedance for that frequency, just as in the case of the open-circuited line.

The results obtained in this analysis are illustrated in Fig. 84. Curve 1 is the impedance-frequency curve of a uniform line free from any irregularity. Curve 2 is the impedance-frequency curve of the open-circuited line. Curve 3 is the impedance-frequency curve of the short-circuited line.

It will be observed that the impedance-frequency curves of the open-circuited and short-circuited lines are similar in shape though reversed, provided that the open circuit and the short circuit are at the same distance from the sending end. If the impedance at that point differs in any way from the characteristic line impedance, the impedance-frequency curve will be of the same shape, *i.e.*, the sending-end impedance will pass through maxima and minima at definite values of frequency. The values of the maxima and minima will depend, however, upon the severity of the irregularity, being greatest either on open circuit or on short circuit.

The results obtained from this analysis indicate that, if the impedance-frequency curve is obtained experimentally and is wavy

in shape, it is a certain indication that some irregularity exists at some point along the line, though the nature of the irregularity is not at all disclosed by the shape of the curve.

While it may be possible, though quite difficult, to design and construct an artificial network to balance a line having some irregularity, it is far better and easier to design and construct

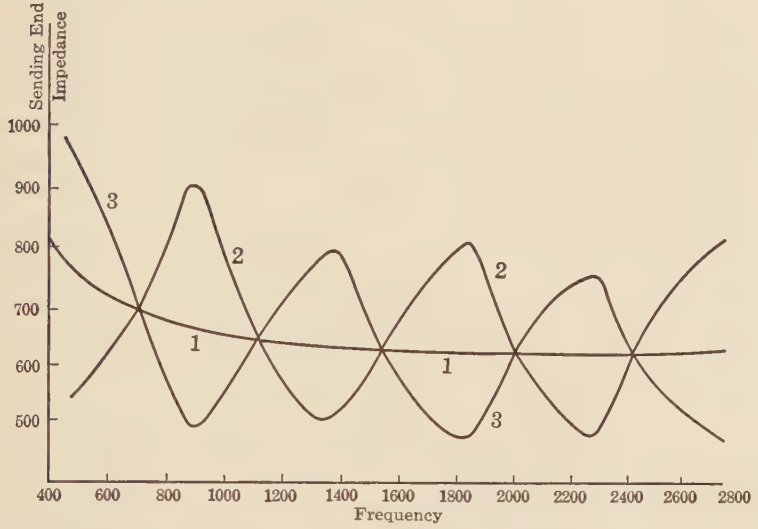

Fig. 84. Variation of Sending-end Impedance with Frequency (Calculated)
Curve 1. Uniform Line closed through Network equal to Z_0.
Curve 2. Open Circuit, 100 Miles from Sending End.
Curve 3. Short Circuit, 100 Miles from Sending End.

balancing artificial networks with impedance-frequency characteristics equal to that of homogeneous, uniform lines, and to maintain the lines free of any irregularities by frequent and systematic impedance-frequency tests.

140. Measurement of Line Impedance by Open-Circuit and Short-Circuit Tests. — Line impedance at any frequency is generally understood to be the characteristic impedance at the frequency under consideration, for it is also the impedance of the line of any length whatever, when terminated by an impedance equal to the characteristic impedance. Under uniform conditions, it may be calculated most readily for any frequency from the electrical line-constants per mile of line.

The actual measurement of the characteristic impedance or line impedance, as defined above, at any frequency, may be carried out by means of the Wheatstone bridge, a schematic diagram of which is shown in Fig. 85.

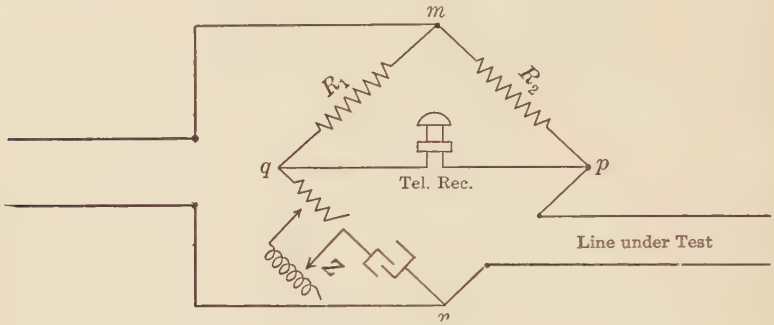

FIG. 85

An alternating potential of adjustable frequency, obtained from a Vreeland oscillator or from a thermionic oscillator, is impressed across the terminals m and n. The ratio arms R_1 and R_2 are equal non-inductive resistances. Z is the variable impedance arm, whose resistance and reactance, for any desired frequency within the range of speech frequencies, may be adjusted to balance the line impedance Z_L, the line itself being the arm n–p of the bridge. A sensitive telephone receiver connected across the terminals p–q is used instead of the ballistic galvanometer ordinarily employed in direct-current bridge measurements. When no sound is heard in the receiver, the impedance Z in arm n–q balances the line impedance in arm n–p. Under these conditions, we see from the nature of the connections shown in the figure that

$$\frac{R_1}{R_2} = \frac{Z}{Z_L}.$$

Since $R_1 = R_2$, it follows that

$$Z = Z_L,$$

where Z_L is the measured line impedance.

141. Calculation of the Characteristic Impedance and the Characteristic Phase Angle from Test Data. — Suppose that such an impedance test at any one desired frequency is made, first with the line open-circuited at the distant terminal, and then with the line short-circuited. Let us set

Z_1 = bridge impedance balancing the line impedance when the line is open-circuited;

r_1 = adjusted resistance component of the balancing impedance Z_1;

x_1 = adjusted reactance component of the balancing impedance Z_1;

Z_2 = bridge impedance balancing line impedance when the line is short-circuited;

r_2 = adjusted resistance component of balancing impedance Z_2;

x_2 = adjusted reactance component of balancing impedance Z_2.

Then we have, for the balanced condition in the two cases,

$$Z_1\epsilon^{j\theta_1} = \sqrt{r_1^2 + x_1^2}\,\epsilon^{j\theta_1}, \tag{12}$$

and

$$Z_2\epsilon^{j\theta_2} = \sqrt{r_2 + x_2}\,\epsilon^{j\theta_2}, \tag{13}$$

where

$$\theta_1 = \tan^{-1}\frac{x_1}{r_1},$$

and

$$\theta_2 = \tan^{-1}\frac{x_2}{r_2},$$

θ_1 being the line phase angle when the line is open-circuited, and θ_2 the line phase angle when the line is short-circuited. But the impedance of an open-circuited line, by (3) Chapter VI, is

$$Z_1 = \frac{Z_0}{\tanh pS}, \tag{14}$$

and that for a short-circuited line, by (3), Chapter VII, is

$$Z_2 = Z_0 \tanh pS. \tag{15}$$

Solving (14) and (15) for $\tanh pS$, and equating the two values, we have

$$\frac{Z_0}{Z_1} = \frac{Z_2}{Z_0},$$

or

$$Z_0 = \sqrt{Z_1 Z_2}. \tag{16}$$

Substituting for Z_1 and Z_2, their values from (12) and (13), respectively, we get

$$Z_0\epsilon^{j\alpha} = \sqrt[4]{(r_1^2 + x_1^2)\,(r_2^2 + x_2^2)}\;\epsilon^{j\frac{\theta_1+\theta_2}{2}}, \tag{17}$$

which gives the characteristic line impedance and its argument, the characteristic phase angle.

142. Calculation of the Propagation Constant, the Attenuation Constant, and the Wave-Length Constant from Test Data. — If the value of Z_0 as measured does not check with the calculated value for the test frequency, there is evidently some irregularity at some point along the line. A calculation of the line constants from the measured characteristic impedance will reveal the kind of irregularity. Thus, solving equations (14) and (15) for Z_0, and equating the two values, we obtain

$$Z_1 \tanh pS = \frac{Z_2}{\tanh pS},$$

whence

$$\tanh pS = \sqrt{\frac{Z_2}{Z_1}}. \tag{18}$$

Using the exponential equivalent for $\tanh pS$, we may write (18) in the form

$$\frac{\epsilon^{pS} - \epsilon^{-pS}}{\epsilon^{pS} + \epsilon^{-pS}} = \sqrt{\frac{Z_2}{Z_1}}.$$

Multiplying both numerator and denominator of the left member of this equation by ϵ^{pS}, we have

$$\frac{\epsilon^{2pS} - 1}{\epsilon^{2pS} + 1} = \sqrt{\frac{Z_2}{Z_1}}.$$

Solving this last equation for ϵ^{2pS}, we get

$$\epsilon^{2pS} = \frac{1 + \sqrt{\frac{Z_2}{Z_1}}}{1 - \sqrt{\frac{Z_2}{Z_1}}}. \tag{19}$$

Substituting for Z_1 and Z_2, their values as given by equations (12) and (13), respectively, we find

$$\epsilon^{2pS} = \frac{1 + \sqrt[4]{\frac{r_2^2 + x_2^2}{r_1^2 + x_1^2}}\, \epsilon^{j\frac{\theta_2 - \theta_1}{2}}}{1 - \sqrt[4]{\frac{r_2^2 + x_2^2}{r_1^2 + x_1^2}}\, \epsilon^{j\frac{\theta_2 - \theta_1}{2}}}. \tag{20}$$

If we set, for brevity,

$$\sqrt[4]{\frac{r_2^2 + x_2^2}{r_1^2 + x_1^2}} = A \text{ and } \frac{\theta_2 - \theta_1}{2} = \theta_3,$$

equation (20) may be written in the abbreviated form

$$\epsilon^{2pS} = \frac{1 + A\epsilon^{j\theta_3}}{1 - A\epsilon^{j\theta_3}}.$$

Using the complex equivalent, this equation may be written in the form

$$\epsilon^{2pS} = \frac{1 + A\cos\theta_3 + jA\sin\theta_3}{1 - A\cos\theta_3 - jA\sin\theta_3},$$

or

$$\epsilon^{2pS} = \frac{\sqrt{(1 + A\cos\theta_3)^2 + A^2\sin^2\theta_3}\,\epsilon^{j\theta_4}}{\sqrt{(1 - A\cos\theta_3)^2 + A^2\sin^2\theta_3}\,\epsilon^{j\theta_5}}, \tag{21}$$

or, finally,

$$\epsilon^{2pS} = B\,\epsilon^{j\theta_6}, \tag{22}$$

where

$$B = \sqrt{\frac{(1 + A\cos\theta_3)^2 + A^2\sin^2\theta_3}{(1 - A\cos\theta_3)^2 + A^2\sin^2\theta_3}}, \tag{23}$$

$$\theta_4 = \tan^{-1}\frac{A\sin\theta_3}{1 + A\cos\theta_3}, \tag{24}$$

$$\theta_5 = \tan^{-1}\frac{-A\sin\theta_3}{1 - A\cos\theta_3}, \tag{25}$$

and

$$\theta_6 = \theta_4 - \theta_5.$$

To find the value of the propagation constant, equation (22) may now be written in the form

$$2\,pS\log_{10}\epsilon = \log_{10}B + j\theta_6\log_{10}\epsilon,$$

whence

$$p = \frac{\log_{10}B}{.4249 \times 2\,S} + j\frac{\theta_6}{2\,S}. \tag{26}$$

Since $p = a + jv$, it follows from (26) that the attenuation constant, as measured, is

$$a = \frac{\log_{10}B}{.8498\,S} \quad \text{hyperbolic radians}, \tag{27}$$

and the wave length constant as measured is

$$v = \frac{\pi\theta_6}{2 \times 180\,S} = .0008722\,\theta_6 \quad \text{circular radians}. \tag{28}$$

Using these values of a and v, the propagation constant may now be calculated from the equation

$$p = \sqrt{a^2 + v^2}\,\epsilon^{j\delta}, \tag{29}$$

where

$$\delta = \tan^{-1}\frac{v}{a}.$$

143. Calculation of R, L, C, and G from Test Data. — If the general expression for the characteristic impedance of a line,

$$Z_0 \epsilon^{j\alpha} = \sqrt{\frac{R + jL\omega}{G + jC\omega}},$$

is multiplied by the general expression for the propagation constant of the same line,

$$p \epsilon^{j\delta} = \sqrt{(R + jL\omega)(G + jC\omega)},$$

it will be seen that their product is

$$Z_0 \epsilon^{j\alpha} p \epsilon^{j\delta} = R + jL\omega. \tag{30}$$

Since both the characteristic impedance and the propagation constant are known by calculation from equations (17) and (29), respectively, it follows that

$$R + jL\omega = pZ_0 [\cos(\alpha + \delta) + j \sin(\alpha + \delta)],$$

whence

$$R = pZ_0 \cos(\alpha + \delta) \quad \text{ohms resistance per mile of line,} \tag{31}$$

and

$$L\omega = pZ_0 \sin(\alpha + \delta) \quad \text{ohms reactance per mile of line,}$$

or

$$L = \frac{pZ_0 \sin(\alpha + \delta)}{2\pi f} \quad \text{henries per mile of line,}$$

where f is the frequency used in the test.

By referring again to the general expressions for the propagation constant and the characteristic line impedance, it will be observed that their ratio is

$$\frac{p \epsilon^{j\delta}}{Z_0 \epsilon^{j\alpha}} = G + jC\omega, \tag{33}$$

whence

$$G + jC\omega = \frac{p}{Z_0} [\cos(\delta - \alpha) + j \sin(\delta - \alpha)],$$

from which it follows that

$$G = \frac{p}{Z_0} \cos(\delta - \alpha) \quad \text{mhos per mile of line,} \tag{34}$$

and

$$C\omega = \frac{p}{Z_0} \sin(\delta - \alpha) \quad \text{mhos susceptance per mile of line,}$$

or

$$C = \frac{\frac{p}{Z_0} \sin(\delta - \alpha)}{2\pi f} \quad \text{farads per mile of line.} \tag{35}$$

144. Location of Irregularity. Theoretical Considerations. — It was shown in § 139 that if the impedance-frequency curve for any line, when obtained experimentally by some method like the one outlined in the previous sections, is not a smooth curve but wavy in shape, there is some irregularity at some point along the line, which must be located and removed in order that amplification be successful.

That the impedance-frequency curve obtained experimentally enables us to locate the irregularity existing at some point along the line, may be shown in the following manner. Suppose that the irregularity is an open circuit. From what has been said in § 139, the measured impedance will pass through maximum values when the frequency is (see (5))

$$f = \frac{nV}{2S}.$$

Hence, if f_1, f_2, f_3, etc., are the frequencies for successive maxima, it follows that

$$\frac{2Sf_1}{V} = n_1, \quad \frac{2Sf_2}{V} = n_1 + 1, \quad \frac{2Sf_3}{V} = n_1 + 2, \text{ etc.},$$

or, rearranging,

$$n_1 = \frac{2Sf_1}{V}, \; n_1 = \frac{2Sf_2}{V} - 1, \quad n_1 = \frac{2Sf_3}{V} - 2, \text{ etc.}$$

Hence, by equating any successive pair, and solving for S, which is the distance from the testing end to the irregularity, we get

$$(36) \qquad S = \frac{V}{2(f_2 - f_1)}; \quad S = \frac{V}{2(f_3 - f_2)}, \qquad \text{etc.}$$

This shows that the distance to the irregularity, which in this particular illustrative case, in an open circuit, can be obtained by dividing the velocity of propagation by twice the *frequency interval* between successive maxima. Assuming now that the irregularity is a short circuit, we see that the measured sending-end impedance will pass through maxima (see (7)) when

$$f = \frac{(2n - 1)V}{4S}.$$

Hence, if f_1, f_2, f_3 are the frequencies for successive maxima, it follows that

$$\frac{4\,Sf_1}{V} = 2\,n_1 - 1,$$

$$\frac{4\,Sf_2}{V} = 2\,(n_1 + 1) - 1 = 2\,n_1 + 1,$$

$$\frac{4\,Sf_3}{V} = 2\,(n_1 + 2) - 1 = 2\,n_1 + 3,$$

etc.; or, rearranging,

$$2\,n_1 = \frac{4\,Sf_1}{V} + 1, \qquad 2\,n_1 = \frac{4\,Sf_2}{V} - 1, \qquad 2\,n_1 = \frac{4\,Sf_3}{V} - 3, \text{ etc.}$$

Equating successive pairs, and solving for S, which is the distance to the irregularity measured from the sending end, we get

$$(37) \qquad S = \frac{V}{2\,(f_2 - f_1)}, \qquad S = \frac{V}{2\,(f_3 - f_2)}, \text{ etc.}$$

Since the expression for calculating the distance S to the irregularity as seen from equation (36) and (37), is the same for the extreme types of irregularities (open circuit and short circuit) that may be encountered, we may infer that the same expression will also hold for any kind of irregularity between the limits of open circuit and short circuit.

The location of any irregularity consists, then, in obtaining experimentally the impedance-frequency curve, and dividing the velocity of propagation by twice the frequency interval between maxima. It is obvious, however, that since the velocity of propagation changes with the frequency, and since the velocity was assumed constant in the preceding method for calculating the distance to the irregularity, the calculated value of S will be only approximately correct. Knowing, however, the approximate location of the irregularity, the exact location may then be obtained by a process of elimination, explained at the end of § 146.

145. Line Impedance Measurements, Practical Considerations.— The two things in view in making line impedance measurements are (1) the determination of line conditions, whether the line is free or not of any irregularity, and (2) the location and removal of the irregularity, if any. It is important, therefore, that the line be

disconnected from phantom repeating coils, composite sets, or any other apparatus that would tend to change the assumed uniform distribution of line constants, since these might change the impedance and thus overshadow the irregularity that is still unknown and is to be discovered.

The line is closed at its far end through an artificial network, designed and constructed to have an impedance-frequency characteristic curve similar to that of the line assumed uniform, *i.e.*, free of any irregularities. The line is thereby made equivalent in behavior to an infinitely long one, and the impedance as measured by means of the Wheatstone bridge will then be equal to the characteristic impedance (should the line be uniform) at test frequency.

When measuring the impedance of loaded lines, it is also imperative that terminal conditions should be such as to avoid errors due to asymmetry, *i.e.*, if the line ends at its testing terminus with a full loading coil or a half loading coil, the artificial network at the other terminus should also begin with a full or a half loading coil, respectively.

Since, as stated elsewhere, line leakage conductance varies with weather conditions, a line insulation test is essential before making impedance measurements. If the tested insulation resistance is less than 5 megohms ($G = .2$ micromhos) the line is not in condition for the impedance test, which must be postponed until more favorable weather conditions prevail.

Actual values of line impedance within the range of speech frequencies are not absolutely essential in the determination of line conditions. We may consider the impedance at any one frequency as being the resultant of an ohmic resistance component comprising the series resistance of the line and the parallel leakage conductance combined, and an equivalent reactive component comprising the series inductive reactance and the parallel capacity susceptance also combined. It will be apparent that tedious calculations can be avoided by obtaining experimentally the variation of these components with the frequency, and using instead of the impedance-frequency curve, either the equivalent resistance-frequency curve, or the equivalent reactance-frequency curve in determining line conditions and in locating irregularities.

If the line phase angle α is positive for all values of frequency, the balancing arm of the Wheatstone bridge need contain only a variable resistance and a variable inductance. These when ad-

justed to balance the line impedance, give directly the values of the equivalent line resistance R and equivalent line inductance L.

On the other hand, if the phase angle α is negative, indicating that the effective line reactance is dielectric in character, inductance may be switched in, in series with the line, and its value may be adjusted until no sound is heard in the receiver. It is evident that under this condition the inductive reactance introduced in series with the line is just equal to the equivalent line capacity reactance: $L\omega = 1/C\omega$. It follows, therefore, that the variable impedance arm of the bridge need not contain any variable capacitance at all, but only a variable inductance, which must, however, be connected in such a manner that it can be switched in, either in series with the line, in which case it is conventionally a negative reactance $-1/C\omega$, or in series with the balancing impedance arm, in which case it is conventionally a positive reactance $L\omega$.

In either of the above conditions, the inductance, whether in series with the balancing arm, or in series with the line under test, as the case may be, must be multiplied by $2\pi f$ to give the equivalent reactance. It is customary, in practical line impedance tests, to dispense even with this calculation and to plot not the equivalent reactance, but only the measured inductance L against the frequency, marking it $+L$ when in series with the variable arm of the impedance bridge, and $-L$ when in series with the line under test. It must be borne in mind, however, that $-L$ has no physical meaning whatever, it being only the mathematical equivalent of $-1/C\omega^2$, and that it is used only as an expedient to save time.

146. The Impedance Bridge. — The bridge used for line impedance tests, a schematic diagram of which is given in Fig. 86, consists of the two ratio arms of non-inductive resistances of 1000 ohms each, the variable impedance arm consisting of a dial rheostat with an adjustable resistance from 0 to 10,000 ohms, and an adjustable inductance, commonly called an *inductometer*.

To cover the entire range of values of inductance needed for line impedance tests, as well as of apparatus or networks, the impedance bridge set includes two inductometers, one having a range of from 20 to 110 millihenries and the other a range of from 100 to 400 millihenries. Since inductometers cannot be constructed to begin sharply at L equal to 0, the testing set includes also a fixed inductance of 100 millihenries, which, when necessary, may be switched in

opposition to either one of the inductometers mentioned above. It is obvious that the total inductance in the circuit is the adjusted one less the fixed value of 100 milihenries.

In switching in an adjustable inductance, alone or in opposition to the fixed inductance mentioned above, in either arm of the impedance bridge, to bring about a balance with the line impedance, the resistance naturally associated with the inductance coil is also connected in that arm of the bridge. For proper balance, it is necessary that a compensating equal resistance be connected in the opposite bridge arm. For this purpose the bridge set includes compensating resistances for each of the two inductometers, and also for the fixed inductance of 100 henries. These are switched in automatically in the arm opposite to that containing for the time being the inductance used. All these requirements are embodied in the impedance bridge set shown schematically in Fig. 86.

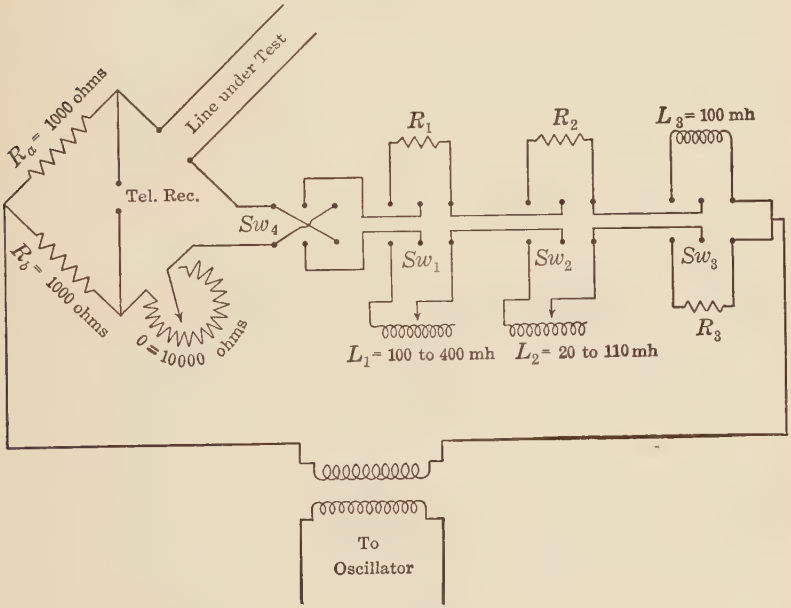

FIG. 86

In taking impedance-frequency measurements, the frequency of the supply is adjusted to that required for the test, and the current is also adjusted to a value that would give a tone of sufficient strength in the receiver. If we assume that the line phase angle is

negative, which is the usual condition, especially with non-loaded lines, the procedure in making a test is as follows. Close the switch Sw_4 to the right, thereby connecting inductance in series with the line. In this case, the adjusted inductance to obtain a balance is conventionally negative. The switch Sw_1 is then closed to the left, thereby connecting the inductometer L_1 and its resistance in series with the line. The compensating resistance R_1 for this inductometer is automatically thrown in in series with the adjustable resistance in the balancing arm of the bridge. With the switches Sw_2 and Sw_3 now closed to the right, the inductometer L_2 and its compensating resistance R_2, and also fixed inductance L_3 and its associated resistance R_3, are entirely out of the circuit.

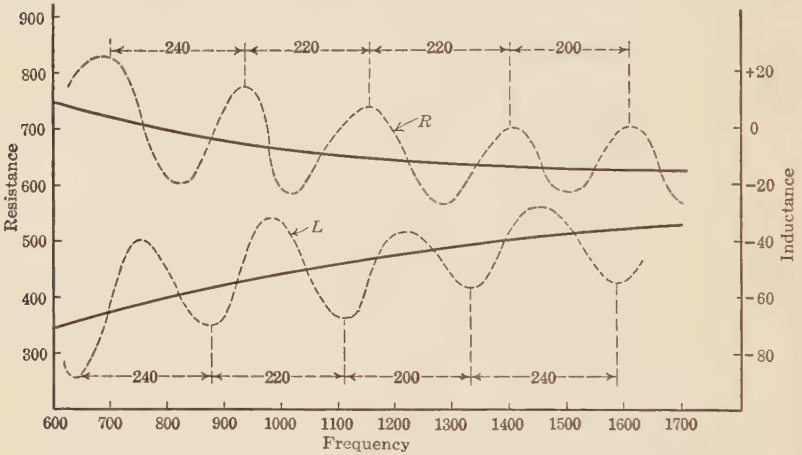

Fig. 87. Variation of Equivalent R and L with Frequency. Line Impedance Test. Non-loaded Line 460 Miles Long

If Sw_2 is closed to the left, and Sw_3 to the right, the inductometer L_2 and its compensating resistance R_2 are in series with inductometer L_1 and its compensating resistance R_1, respectively, while L_3 and R_3 are out. If Sw_3 is closed to the left, L_3 is in opposition to the inductometer, and R_3 in opposition to L_3 to compensate for its resistance. Various other combinations can be made to bring about a balancing condition, which is finally obtained by adjusting both adjustable resistance and inductance until no sound is heard by the observer listening to the telephone receiver connected as shown in the figure. In this manner the equivalent ohmic resist-

ance and equivalent inductance of the line are obtained for the range of speech frequencies beginning with about 300 cycles, in steps of about 50 cycles, to about 2400 cycles per second. The values of R and L obtained on balance are then plotted against corresponding frequencies.

Figure 87 shows the *resistance-frequency* and *inductance-frequency* curves for a 460-mile non-loaded line tested with its far end closed through a network whose impedance-frequency curve corresponds to that of the line assumed free of irregularities. It will be noted that the average frequency interval between successive maxima is 226 cycles. Taking the velocity of propagation as 180,000 miles per second, the approximate distance from the testing end to the irregularity, by (37), is

$$S = \frac{180\ 000}{2 \times 226} = 398 \text{ miles.}$$

Figure 88 gives the resistance-frequency and the inductance-frequency curves for a regularly series-loaded line 170 miles long, closed

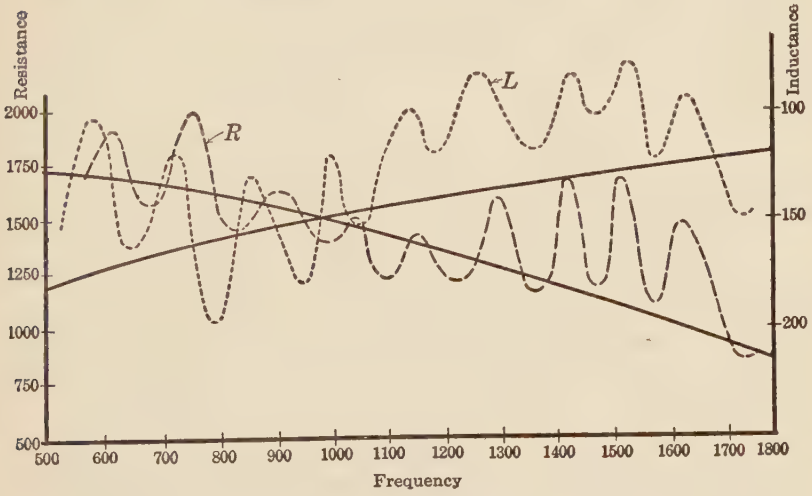

FIG. 88. Variation of Equivalent R and L with Frequency. Line Impedance Test. Loaded Line 170 Miles Long

through an artificial network whose impedance frequency characteristic curve is similar to that of this line assumed free of any irregularity. The average frequency interval is 126 cycles. Taking the

velocity of propagation for loaded lines as 56,000 miles per second, the approximate distance to the irregularity is

$$S = \frac{56\ 000}{2 \times 126} = 222 \text{ miles.}$$

Since this is more than the length of the line, it is evident that the trouble is with the terminal network.

Figure 89 shows the resistance-frequency and the inductance-frequency curves of a loaded line 190 miles long. The average frequency interval is 167 cycles, and the approximate distance to the

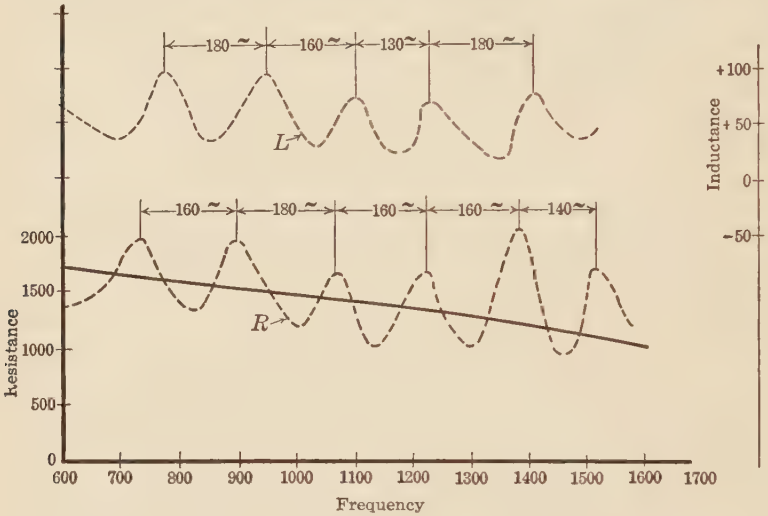

Fig. 89. Variation of Equivalent R and L with Frequency. Line Impedance Test. Loaded Line 190 Miles Long

irregularity is 168 miles. Taking 8 miles as the spacing distance between loading coils, the irregularity appears to be with the 21st loading section.

As stated in § 144, by assuming the velocity of propagation as constant for all frequencies, 180,000 miles per second for non-loaded lines, and 56,000 miles per second for loaded lines, the calculated distance to the irregularity is only approximately correct. The actual location of the trouble is obtained by repeating the test with the line closed through a portable network at the point determined by calculation. If this test shows the line free of any irregularity, it is obvious that the trouble is beyond this

point. The test is then repeated at the next loading section and so on until the exact location is finally obtained.

147. Balancing Terminal Networks. — As explained before, the terminal network employed to balance a line for the successful operation of amplifiers must have the same impedance-frequency characteristic curves as the uniform line. This means that the equivalent resistance and equivalent reactance of this network shall change with the frequency in the same manner as the equivalent resistance component and equivalent reactive component of the line, as shown by the smooth curves in Figs. 87, 88, and 89.

It should be clearly understood that a balancing network is *not* an artificial equivalent of the line in the same sense as the equivalent T or Π circuits discussed in Chapter XI, since the propagation constant of the line is of no consideration in the design and construction of a balancing network. An *artificial line* is a network equivalent to the line with regard to its transmission characteristics, both impedance and propagation constant, but for only one frequency, the one for which it is designed; while a *balancing-network* has only the same impedance-frequency characteristic curve as the line. A balancing network, designed for a line of definite specifications, will balance any length of line, while a T or Π equivalent circuit will be the equivalent of a line of definite length, viz., the length for which it has been designed.

It will be evident from what precedes that standard balancing networks can be constructed for use with any line of similar specifications with regard to electrical constants and loading-coil impedances. Such networks are generally called *basic networks*, for they are designed to balance lines with regular termination: no cable at either terminal for open aerial non-loaded lines; loaded lines with first loading coil placed at a distance from the sending end equal to .2 of the spacing distance between the loading coils; loaded lines terminated with half loading coils or mid-loading coil section.

For lines with such regular termination, the basic network is rather simple in construction, consisting of a resistance in parallel with a condenser, the values of which may be calculated from the characteristic impedance of the line at the average speech frequency of 796 cycles per second.

The characteristic admittance of the line, at that particular frequency, will be

$$Y_0 = \frac{1}{Z_0}\epsilon^{-j\alpha} = \frac{1}{Z_0}(\cos\alpha - j\sin\alpha).$$

This must also be the admittance of the parallel balancing network, whose resistance branch is

$$r = \frac{Z_0}{\cos\alpha}, \tag{38}$$

and whose susceptance branch is

$$C\omega = \frac{\sin\alpha}{Z_0}. \tag{39}$$

Hence the condenser must have a capacitance of

$$C = \frac{\sin\alpha}{5000\,Z_0}. \tag{40}$$

With the resistance and capacitance of the two parallel branches as calculated by equations (38) and (40), the admittance of the network at any frequency whatever will be

$$Y = \frac{1}{r} + jC\omega,$$

and its impedance will be

$$Z = \frac{\frac{1}{r}}{\frac{1}{r^2} + C^2\omega^2} - j\,\frac{C\omega}{\frac{1}{r^2} + C^2\omega^2}. \tag{41}$$

From this it is evident that both equivalent resistance and equivalent reactance of the balancing network will vary with the frequency along smooth curves.

If an aerial non-loaded line is connected to the amplifying circuit through a short length of cable, the balancing network must include the electrical equivalent of the cable. If the loop resistance of this cable is small in comparison with its loop reactance, a condenser of a capacitance equal to that of the cable in series with the basic network will usually give sufficient balance. With loaded lines, the balancing network must include, in addition to the basic network, the equivalent impedance of either half-section, or half loading coil, depending on how the line is terminated, or the equivalent impedance of any apparatus connected between the line and amplifier output terminals.

CHAPTER XVII

TRANSMISSION EQUIVALENTS

148. The Standard Cable; the Standard Mile. — The ***transmission ability*** of a telephone line may be determined either in terms of its electrical current efficiency, or in conjunction with its terminal sound-converting apparatus from the point of view of commercial speech transmission. The first takes cognizance only of the attenuating properties of the line, and it may be obtained by calculation. The second, which is usually called the ***telephonic value,*** takes into consideration all the line properties that affect the transmission of articulated speech, and can be determined only by actual test.

Since the loudness or volume of a sound is directly proportional to the amplitude of the corresponding current wave, it follows that the ratio of the current at the receiving end to that at the sending end is a measure of the volume of the received sound in percentage of the volume at the sending end. The current efficiency may therefore be taken as a direct measure of the effectiveness with which a line will permit the propagation of a train of waves of single frequency. The receiving-end current and the sending-end current for any line of any length and with any termination whatever may be calculated by the formulas developed in Chapter VIII, and thereby predetermine its transmitting ability or change in transmitting ability with a change in length or terminal conditions.

It is customary, however, in practical telephony, to express the transmitting ability of lines, not in terms of the current efficiency, but in terms of the number of miles of an arbitrarily chosen line, commonly referred to as the ***standard cable.*** One mile of this standard cable is called a ***standard mile*** and is generally denoted by the symbol M. S. C.*

* The artificial standard cable used in testing is a representation of this No. 19 standard-gauge cable. It has 88 ohms resistance and .054 mf. capacitance per loop mile, with no leakance and no inductance. The attenuation constant of this artificial cable is .109 hyperbolic radians at 800 cycles per second.

This arbitrarily chosen standard line is the No. 19 B. & S. non-loaded, paper-insulated cable, the transmission constants of which, as given in Table 1, are:

$$Z_0 = 571\,\epsilon^{-j42.7^\circ} \text{ ohms}$$
$$a = .107 \text{ hyperbolic radians}$$
$$v = .111 \text{ circular radians.}$$

It must be fully understood that this line is not a *standard* from the point of view of excellent transmitting qualities. As a matter of fact, the No. 19 cable has a high attenuation constant, resulting in low efficiency, and the relative values of its electrical constants are such that the line is far from distortionless. The maximum distance over which articulated speech can be transmitted with distinctness when this line is terminated with standard subscriber's apparatus is only about 30 miles.

The name *standard cable* should be understood to mean merely a *reference line,* in terms of which the transmitting abilities of other lines of any length and termination whatever may be expressed most conveniently. In this light, it will be understood readily that *one mile of standard cable* is merely a unit in which attenuation losses during transmission, or transition losses or gains due to terminal reflections, or due to various apparatus bridged across or inserted in series, may be expressed either when predetermined by calculation or measured experimentally.

149. Transmission Equivalents of Lines Free of Terminal Reflections. — A line of length S miles, having an attenuation constant equal to a hyperbolic radians, closed through a terminal receiving apparatus of impedance equal to the characteristic line impedance, is said to be equivalent in transmitting ability to a standard cable of length S_c miles closed through an impedance equal to its characteristic line impedance, when the receiving-end current and the sending-end current of the two lines, respectively, are equal to one another.

For lines with terminals of this character, the current decays exponentially, according to the law

$$I_r = I_s\epsilon^{-aS}.$$

It follows that when the receiving-end current and the sending-end current of the two lines are equal to one another, the attenuation will be the same for the two lines:

(1) $$\epsilon^{-.107S_c} = \epsilon^{-aS},$$

or

(2) $$.107\, S_c = aS.$$

This relation shows that a line free from terminal reflections is equivalent to a standard line also free from terminal reflections, when their respective attenuation factors are equal. It is obvious that we may write (2) in the form

(3) $$\frac{.107}{a} = \frac{S}{S_c},$$

which indicates that *one mile of standard cable* having an attenuation constant of .107 hyperbolic radians, is equivalent in its transmitting ability, in other words, will cause attenuation in the same proportion as $.107/a$ miles of line having an attenuation constant of a hyperbolic radians. The ratio of the attenuation constant of the standard cable to the attenuation constant of the line under consideration is called *transmission equivalent* of the standard in terms of the line considered. Obviously, its reciprocal is the *transmission equivalent of the line* in terms of the standard cable. Hence it represents the length of standard cable equivalent in attenuation to one mile of line under consideration.

The following table gives the calculated values of the transmission equivalents of various lines commonly used in telephonic transmission, in terms of the standard cable free of terminal reflections: *

These transmission equivalents of lines give only a general, and often a misleading, idea of their transmitting abilities. They were calculated on the assumption that the standard and the equivalent lengths of the other lines have equal currents at the sending end for equal currents at the receiving end. But since the sending-end impedance of lines free from terminal reflections is equal to the characteristic impedance of the respective lines it follows that if the currents at the sending end are to be equal, the voltage at the sending end of each line, *i.e.*, the transmitting instrument, must be a different one for each case.

* The transmission equivalents used by the Bell System obtained experimentally by the method described in § 153, this Chapter, differ somewhat from those given in this table because the attenuation constant of the artificial cable in the transmission measuring set is .109 hyperbolic radians.

Thus if we say that the transmission equivalent of a No. 16 cable ($Z_0 = 346\ \epsilon^{-j41.4°}$) is 1/1.244 M. S. C., and that of a No. 10 B. & S. aerial, non-loaded copper line ($Z_0 = 723\ \epsilon^{-j11.1°}$) is 1/11.733 M. S. C., meaning by that that one mile of standard cable is equivalent to or will produce the same attenuation in current as 1.244 miles of No. 16 cable or 11.723 miles of No. 10 aerial line, it must be kept in mind that with a given sending-end voltage E_s of standard frequency for the standard cable, the sending-end voltage for the No. 10 line must be (723/571) E_s or 44.13 per cent higher, and that for the No. 16 cable must be (346/571) E_s or 39.41 per cent lower than that for the standard cable.

TABLE 10

GAUGE	DIAMETER IN MILLS	TYPE OF LINE	a	TRANS. EQUIV. in M. S. C.
24 B. & S.*	20.1	Dry core insulated cable	.242	1/.442
22 B. & S.*	25.55		.1897	1/.555
22 B. & S.	25.55		.1743	1/.613
19 B. & S.	38.89		.127	1/.842
19 B. & S.†	38.89		.107	1/1.000
16 B. & S.	50.82		.086	1/1.244
14 B. & S.	64.08		.077	1/1.374
13 B. & S.	71.96		.0575	1/1.861
10 B. & S.	101.9		.0366	1/2.923
14 B. & S.	64.08	Aerial, bare copper clad 47%	.0308	1/3.470
12 B. & S.	80.81		.02305	1/4.637
10 B. & S.	101.9		.01695	1/5.779
9 B. & S.	114.4		.01455	1/7.365
12 N. B. S.	104.0	Aerial, bare copper	.00916	1/11.681
10 B. & S.	101.9		.009119	1/11.733
9 B. & S.	114.4		.00822	1/12.652
8 N. B. S.	160.0		.00628	1/17.038
8 B. W. G.	165.0		.0048	1/22.310

* .084 mf. capacitance. † .054 mf. capacitance.

From a practical point of view, it would appear, therefore, that the standard cable free from terminal reflections under the same conditions of sending-end voltage and frequency, *i.e.*, with similar transmitting apparatus as the lines under consideration, would be a more justifiable reference standard than the one under the same conditions of sending-end current.

150. Nominal Line Admittance as a Measure of the Transmitting Ability of a Line. — Consider a line of characteristic impedance Z_0 and propagation constant p, of any length S miles, terminated with any impedance Z_r. By equation (13), Chapter 8, the current at the receiving end of such a line is

$$I_r = \frac{E_s}{Z_r \cosh pS + Z_0 \sinh pS}.$$

Using the exponential equivalents for the hyperbolic terms, we may write this expression in the form

$$\frac{I_r}{E_s} = \frac{2\,\epsilon^{-pS}}{Z_r\,(1 + \epsilon^{-2pS}) + Z_0\,(1 - \epsilon^{-2pS})}. \tag{4}$$

This ratio of the receiving-end current to the sending-end voltage is of the nature of an admittance. Its numerical value is therefore a direct measure of the current at the receiving end under any conditions of sending-end voltage, line-length, or receiving-end impedance. It may, therefore, be considered also as a measure of the transmitting ability of the line. It will be referred to hereafter as the *nominal line admittance*, and will be denoted by the symbol Y_n.

If the impedance at the receiving end is equal to the characteristic line impedance, the line being free from terminal reflections ($Z_r = Z_0$), the preceding expression for the nominal line admittance simplifies to the form

$$\frac{I_r}{E_s} = \frac{1}{Z_0}\,\epsilon^{-pS}. \tag{5}$$

The maximum nominal line admittance, and hence the maximum receiving-end current for such a line, clearly obtains when the receiver is directly connected to the transmitting apparatus, *i.e.*, when S is equal to zero. For any other line-length, with a termination of such a character, the nominal line admittance, and hence the receiving-end current, will be a decaying exponential function, and may be expressed in per cent of the maximum obtainable.

The maximum nominal line admittance, and hence the maximum receiving-end current of a line terminated with a receiver of impedance Z_r, is obviously $1/Z_r$. It is obtained when the receiving instrument is connected directly to the transmitting apparatus ($S = 0$). The nominal line admittance for any other line length

may be expressed as in the previous case, in terms of the maximum obtainable. For this purpose, expression (4) may be written in the form

$$Y_n = \frac{2\,Z_r\epsilon^{-pS}}{Z_r\,(1+\epsilon^{-2pS}) + Z_0\,(1-\epsilon^{-2pS})} \quad \frac{1}{Z_r}. \tag{6}$$

151. Transmission Equivalents of Lines in Terms of the Standard Cable Free of Terminal Reflections under Similar Transmitting-End Conditions. — The transmission equivalent of any line, with any receiving-end termination whatever, and definite transmitting apparatus, in terms of standard cable free from terminal reflections, may be defined as the length of standard cable of uniform attenuation whose nominal admittance is equal numerically to that of the line under consideration.

By equation (5) the nominal admittance of the standard cable free from terminal reflections and of length S_c miles is

$$Y_{nc} = \frac{1}{571}\,\epsilon^{-.107S_c}. \tag{7}$$

If Y_n is the nominal admittance of a line of any length whose transmission equivalent is to be calculated, its nominal admittance must be equal to the nominal admittance of the standard cable to which it is equivalent, *i.e.*, $Y_n = Y_{nc}$, or, by (7),

$$Y_n = \frac{1}{571}\,\epsilon^{-.107S_c}, \tag{8}$$

whence

$$S_c = \frac{\log_{10}\dfrac{1}{571\,Y_n}}{.0465}, \tag{9}$$

so that the value of S_c may be obtained from tables of exponential functions. This value of S_c represents the length of standard cable of uniform attenuation, which, with the same transmitting apparatus, will have the same receiving-end current as the line considered. Hence it is its transmission equivalent.

Thus the numerical value of the 100-mile No. 10 B. & S. non-loaded aerial copper line, terminated with an impedance equal to its characterisic line impedance, is, by (5),

$$Y_n = \frac{1}{723}\,\epsilon^{-100\times.009119} = \frac{.4017}{723}\ \text{mhos.}$$

For equivalence with the standard cable, we get

$$\frac{1}{571}\,\epsilon^{-.107S_c} = \frac{.4017}{723},$$

whence, by (9),

$$S_c = \frac{\log_{10}\dfrac{723}{571 \times .4017}}{.0465} = 10.71 \text{ M. S. C.}$$

This line, when regularly series-loaded with impedance coils of $1000\,\epsilon^{j89.6°}$ ohms, spaced 9.64 miles apart, has a characteristic impedance of $Z_0 = 1662\,\epsilon^{j1.6°}$ and an attenuation constant of .004359 hyperbolic radians. The nominal line admittance of this line, when terminated with a receiver of impedance equal to its characteristic line impedance, is

$$Y_n = \frac{1}{1662}\,\epsilon^{-100\times.004359} = \frac{.6472}{1662} \text{ mhos.}$$

Its transmission equivalent in terms of the standard cable free from terminal reflections is, by (9),

$$S_c = \frac{\log_{10}\dfrac{1662}{571 \times .6472}}{.0465} = 14.04 \text{ M. S. C.;}$$

or since

$$\epsilon^{-.107S_c} = \frac{571 \times .6472}{1662} = .223,$$

we find, from tables of exponential functions,

$$.107\,S_c = 1.5, \text{ or } S_c = 14.04 \text{ M. S. C.}$$

This shows a transition loss over the non-loaded line of $14.04 - 10.71 = 3.33$ M. S. C. due to increased characteristic line impedance resulting from loading.

Similarly, the nominal admittance of a 30-mile non-loaded No. 19 B. & S. cable, having .054 mf. capacitance per mile of loop and terminated with a receiver of impedance $300\,\epsilon^{j50°}$ ohms, is, by (4),

$$Y_n = \frac{2\,\epsilon^{-pS}}{300\,\epsilon^{j50°}(1 + \epsilon^{-2pS}) + 571\,\epsilon^{-j42.7°}(1 - \epsilon^{-2pS})},$$

where

$$\epsilon^{-pS} = \epsilon^{-30\times.107} \times \epsilon^{-j30\times.111} = .0404\,\epsilon^{-j190.8°},$$

and

$$\epsilon^{-2pS} = \epsilon^{-60\times.107} \times \epsilon^{-j60\times.111} = .001627\,\epsilon^{-j21.6°}.$$

Neglecting ϵ^{-2pS} as being too small to affect the result, we get

$$Y_n = \frac{.0808\ \epsilon^{-j190.8°}}{300\ \epsilon^{j50°} + 571\ \epsilon^{-j42.7°}},$$

where

$$300\ \epsilon^{j50°} = 192.84 + j219.8,$$

$$571\ \epsilon^{-j42.7°} = 419.63 - j389.25,$$

whence

$$300\ \epsilon^{j50°} + 571\ \epsilon^{-j42.7°} = 612.47 - 68.45\,j = 616\ \epsilon^{-j6.2°}.$$

Substituting this in the preceding expression for the nominal line admittance, we get

$$Y_n = \frac{.0808}{616}\ \epsilon^{-j184.6°}\ \text{mhos.}$$

Hence the transmission equivalent of this line, terminated with a receiving apparatus of impedance $300\ \epsilon^{j50°}$, is, by (9),

$$S_e = \frac{\log_{10} \dfrac{616}{571 \times .0808}}{.0465} = 24.2\ \text{M. S. C.}$$

This result indicates that the 30-mile No. 19 cable terminated with a receiver of impedance $300\ \epsilon^{j50°}$ ohms is equivalent to only 24.2 miles of the same line, when free of terminal reflections. The gain of $30 - 24.2 = 5.8$ M. S. C. is due partly to the smaller receiving-end impedance, and more particularly to the fact that the receiver phase angle is nearly the conjugate of the characteristic phase angle.

The nominal admittance of the 100-mile No. 10 B. & S. non-loaded aerial copper line, terminated with a receiver of impedance $300\ \epsilon^{j50°}$ ohms, is, by (4),

$$Y_n = \frac{2\ \epsilon^{-pS}}{300\ \epsilon^{j50°}\ (1 + \epsilon^{-2pS}) + 723\ \epsilon^{-j11.1°}\ (1 - \epsilon^{-2pS})},$$

where

$$\epsilon^{-pS} = \epsilon^{-100\times.009119}\ \epsilon^{-100\times j.02774} = .4017\ \epsilon^{-j159°},$$

and

$$\epsilon^{-2pS} = \epsilon^{-200\times.009119}\ \epsilon^{-200\times j.02774} = .1614\ \epsilon^{-j318°},$$

Substituting in the above expression, we get

$$Y_n = \frac{.8034\ \epsilon^{-j159^\circ}}{300\ \epsilon^{j50^\circ}\ (1 + .1614\ \epsilon^{-j318^\circ}) + 723\ \epsilon^{-j11.1^\circ}\ (1 - .1614\ \epsilon^{-j318^\circ})},$$

where

$$\begin{aligned} 300\ \epsilon^{j50^\circ} &= 192.8 + j219.8, \\ 300\ \epsilon^{j50^\circ} \times .1614\ \epsilon^{-j318^\circ} &= -\ 1.7 + j48.4, \\ 723\ \epsilon^{-j11.1^\circ} &= 709.6 - j138.9, \\ -\ 723\ \epsilon^{-j11.1^\circ} \times .1614\ \epsilon^{-j318^\circ} &= -\ 100.1 - j59.9 \end{aligned}$$

The denominator in the expression for Y_n is, therefore,

$$800.6 + 70.3\,j = 804\ \epsilon^{j41.3^\circ},$$

whence

$$Y_n = \frac{.8034}{804}\ \epsilon^{-j117.7^\circ}.$$

By (9), the transmission equivalent of this line is

$$S_c = \frac{\log_{10} \dfrac{804}{571 \times .8034}}{.0465} = 5.26 \text{ M. S. C.}$$

Comparing this result with the transmission equivalent of the same line, but with no terminal reflections, it will be noted that there is a transmission gain of $10.71 - 5.26 = 5.45$ M. S. C. due to the fact that the termination differs from the characteristic line impedance in both modulus and argument.

Similarly, the 100-mile No. 10 B. & S. loaded line, terminated with a receiver of impedance $300\ \epsilon^{j50^\circ}$ ohms, by (4), has a nominal admittance of

$$Y_n = \frac{2\ \epsilon^{-pS}}{300\ \epsilon^{j50^\circ}\ (1 + \epsilon^{-2pS}) + 1662\ \epsilon^{j1.6^\circ}\ (1 - \epsilon^{-2pS})},$$

where

$$\begin{aligned} \epsilon^{-pS} = \epsilon^{-100\times.004359}\ \epsilon^{-100\times j.008843} &= .6466\ \epsilon^{-j146.9^\circ}, \\ \epsilon^{-2pS} = \epsilon^{-200\times.004359}\ \epsilon^{-200\times j.008843} &= .4182\ \epsilon^{-j293.8^\circ}, \\ 300\ \epsilon^{j50^\circ} &= 192.8 + j219.8, \\ 300\ \epsilon^{j50^\circ} \times .4182\ \epsilon^{-j293.8^\circ} &= -\ 55.36 + j112.5, \\ 1662\ \epsilon^{j1.6^\circ} &= 1661.00 + j46.3, \\ -\ 1662\ \epsilon^{j1.6^\circ} \times .4182\ \epsilon^{-j293.8^\circ} &= -\ 262.6 + j643.5, \\ & 1535.9 + j1002.1 = 1841\ \epsilon^{j33^\circ}. \end{aligned}$$

Substituting in the preceding equation for the nominal admittance, we get

$$Y_n = \frac{1.2932}{1841}\,\epsilon^{-j113.9^\circ} \text{ mhos,}$$

and the transmission equivalent of this line, terminated with an impedance of $300\,\epsilon^{j50^\circ}$, is, by (9),

$$S_e = \frac{\log_{10}\dfrac{1841}{571 \times 1.2932}}{.0465} = 8.52 \text{ M. S. C.}$$

This indicates a gain over the uniform line of $14.04 - 8.52 = 5.52$ M. S. C. due to the fact that the receiver impedance differs in modulus and in argument from the characteristic line impedance.

152. Transmission Equivalents of Lines in Terms of Standard Cable with Definite Sending-End and Receiving-End Terminations. — The transmission equivalents of lines given in the preceding sections are with reference to the standard cable free from terminal reflections. A more justifiable reference standard from the point of view of telephonic value is clearly a standard cable terminated with a receiver impedance equal to that of the standard receiving instrument commonly used. Such a standard may be thought of as an actual telephone line of variable length, equipped with standard terminal apparatus, both transmitter and receiver. It may be easily constructed, therefore, as an *artificial testing standard*, and as such is called a *transmission measuring set.*

A standard cable terminated with an impedance equal to that of the adapted standard receiver, and having a nominal admittance equal to that of another line equipped with a similar standard receiving instrument, under the same conditions of sending-end voltage and frequency, will have a receiving-end current equal to that of the line under consideration. Under these conditions, the length of standard cable will be equivalent to that of the line, and may be referred to as the ***telephonic transmission equivalent*** of the line, to distinguish it from the transmission equivalent in terms of the standard cable free from terminal reflections. By (4), the nominal admittance of any line of any length whatever is

$$Y_n = \frac{2\,\epsilon^{-pS}}{Z_r\,(1 + \epsilon^{-2pS}) + Z_0\,(1 - \epsilon^{-2pS})}.$$

A standard cable of length S_c equipped with similar transmitting and receiving terminations as the line under consideration, will have a nominal admittance of

$$(10) \qquad Y_{nc} = \frac{2\,\epsilon^{-p_c S_c}}{Z_r\,(1+\epsilon^{-2p_c S_c}) + Z_{0c}(1-\epsilon^{-2p_c S_c})},$$

where the subscript c indicates quantities pertaining to the standard cable.

If Y_n is equal to Y_{nc}, then S_c is the length of standard cable equivalent to the line under consideration. If the value of Y_n is known, either by calculation from the preceding equation, or by actual test, that value may be substituted in equation (10) in place of Y_{nc}, and the resulting equation may be solved for S_c, as follows.

If we set, for brevity,

$$(11) \qquad \epsilon^{-p_c S_c} = X,$$

we may write the equation in the form of a quadratic

$$(12) \qquad Y_n\,(Z_{0c} - Z_r)\,X^2 + 2\,X - Y_n\,(Z_{0c} + Z_r) = 0,$$

whose solution is

$$(13)\ X = \epsilon^{-p_c S_c} = -\frac{1}{Y_n\,(Z_{0c} - Z_r)} \pm \sqrt{\frac{1}{Y_n^2 (Z_{0c} - Z_r)^2} + \frac{Z_{0c} + Z_r}{Z_{0c} - Z_r}}.$$

Assuming that the standard receiving apparatus has an impedance of $300\,\epsilon^{j50^\circ}$ ohms, we have

$$\begin{aligned} Z_{0c} - Z_r &= 571\,\epsilon^{-j42.7^\circ} - 300\,\epsilon^{j50^\circ} = 419.6 - j387.2 - 192.8 - j219.8 \\ &= 226.8 - j\,607.0 = 647.9\,\epsilon^{-j69.6^\circ}, \end{aligned}$$

$$\begin{aligned} Z_{0c} + Z_r &= 571\,\epsilon^{-j42.7^\circ} + 300\,\epsilon^{j50^\circ} = 419.6 - j387.2 + 192.8 + j219.8 \\ &= 612.4 - j\,167.4 = 633.8\,\epsilon^{-j15.3^\circ}, \end{aligned}$$

$$\frac{Z_{0c} + Z_r}{Z_{0c} - Z_r} = \frac{633.8\,\epsilon^{-j15.3^\circ}}{647.9\,\epsilon^{-j69.6^\circ}} = .978\,\epsilon^{j54.3^\circ}.$$

Substituting these values in equation (13), we get

$$\epsilon^{-p_c S_c} = -\frac{1}{Y_n \times 647.9\,\epsilon^{-j69.6^\circ}} \pm \sqrt{\frac{1}{(Y_n \times 647.9\,\epsilon^{-j69.6^\circ})^2} + .978\,\epsilon^{j54.3^\circ}}.$$

To illustrate this method of predetermining the telephonic transmission equivalent of a line, consider the 100-mile No. 10 B. & S. non-loaded line. Its nominal admittance, when terminated with a

receiver of impedance 300 $\epsilon^{j50°}$ ohms, as calculated in the preceding section, is

$$Y_n = \frac{.8034}{804}\,\epsilon^{-j117.7°} \quad \text{mhos},$$

whence

$$Y_n \times 647.9\,\epsilon^{-j69.6°} = \frac{.8034}{804}\,\epsilon^{-j117.7°} \times 647.9\,\epsilon^{-j69.6°},$$

and

$$\frac{1}{Y_n \times 647.9\,\epsilon^{-j69.6°}} = 1.546\,\epsilon^{j187.3°},$$

$$\frac{1}{(Y_n \times 647.9\,\epsilon^{-j69.6°})^2} = (1.546\,\epsilon^{j187.3°})^2 = 2.38\,\epsilon^{j14.6°}.$$

Substituting this value in the preceding equation, we find

$$\epsilon^{-p_c S_c} = -1.546\,\epsilon^{j187.3°} \pm \sqrt{2.38\,\epsilon^{j14.6°} + .978\,\epsilon^{j54.3°}}.$$

But

$$2.38\,\epsilon^{j14.5°} = 2.303 + j\,.600,$$
$$.978\,\epsilon^{j54.3°} = .571 + j\,.794.$$

Hence

$$2.38\,\epsilon^{j14.6°} + .978\,\epsilon^{j54.3°} = 2.874 + j1.394 = 3.178\,\epsilon^{j25.9°}.$$

Substituting this value, we get

$$\epsilon^{-p_c S_c} = -1.546\,\epsilon^{j187.3°} \pm \sqrt{3.178\,\epsilon^{j25.9°}}$$
$$= -1.546\,\epsilon^{j187.3°} \pm 1.782\,\epsilon^{j12.9°}.$$

But

$$-1.546\,\epsilon^{j187.3°} = 1.533 + j.1965,$$
$$1.782\,\epsilon^{j12.9°} = 1.737 + j.397.$$

Hence the two values of $\epsilon^{-p_c S_c}$ are

(*a*) $$\epsilon^{-p_c S_c} = 1.533 + .1965\,j + 1.737 + .397\,j$$
$$= 3.270 + .5935\,j = 3.277\,\epsilon^{j8.6°},$$

(*b*) $$\epsilon^{-p_c S_c} = 1.533 + .1965\,j - 1.737 - 397\,j$$
$$= -.204 - .2005\,j = .286\,\epsilon^{j224.5°}.$$

Since we have

$$\epsilon^{-p_c S_c} = \epsilon^{-.107 S_c}\,\epsilon^{-j.111 S_c},$$

and since we know that $\epsilon^{-.107 S_c}$ cannot be greater than unity, it follows that the only real solution is

$$\epsilon^{-.107 S_c} = .286$$

Hence the value of S_c may be found either from tables of exponential functions or by taking logarithms; it is

$$S_c = \frac{\log_{10} \frac{1}{.286}}{.0465} = 11.26 \text{ M. S. C.}$$

153. Measurement of Telephonic Transmission Equivalents. — The principle underlying the measurement of the telephonic transmission equivalent of a line consists in finding the length of standard cable which, with the same receiving termination and sending-end current, at the average telephonic frequency of 796 cycles per second, would give an equal receiving-end current as the line under test.

It is clear that such a test would not give the *telephonic* value of the line, since articulation or actual speech reproduction is entirely ignored. It will give, however, the telephonic transmission equivalent with respect to volume or loudness. This property, which depends upon the amplitude of the sound wave, is directly proportional to the amplitude of the current wave corresponding to the sound wave producing it, and consequently is also proportional to the effective value as measured by an ammeter at the receiving end of the line. In other words, the equivalence between the line under test and the standard reference cable is only with respect to attenuation.

154. The Transmission Measuring Unit. — The transmission measuring unit developed by the Engineering Department of the American Telephone and Telegraph Company, and used by the Bell System in the measurement of the telephonic transmission equivalents of lines in terms of the standard cable with the standard Bell subscriber's apparatus, as well as the equivalent gains or losses due to line equipment, consists, primarily, of the following parts:

(1) A thermionic oscillator, which supplies the energy at any desired frequency;

(2) A *sending element* located at the terminus of the line under test;

(3) A *receiving element*, located at the other terminus of the line.

Figure 90 shows diagrammatically, but without any details, the connections of the sending element. It will be seen that it consists mainly of a repeating coil T_1, with a 1:2 ratio of transformation, the primary of which is connected to the output terminals of the

oscillator. The circular slide-wire rheostat R_1, in series with the primary, is used to vary the current supplied by the oscillator.

The secondary of this repeating coil, in series with an alternating-current thermal ammeter, is connected across a network R_2R_3. The network R_3 is made up of six non-inductive resistances of 300 ohms each, so connected to a switching arrangement (not shown in the figure) that some of these resistance units may be short-circuited in pairs. In this manner R_3 may be made equal to either 600, 1200, or 1800 ohms, corresponding approximately to the sending-end impedance of (*a*) low-impedance lines such as non-loaded cables

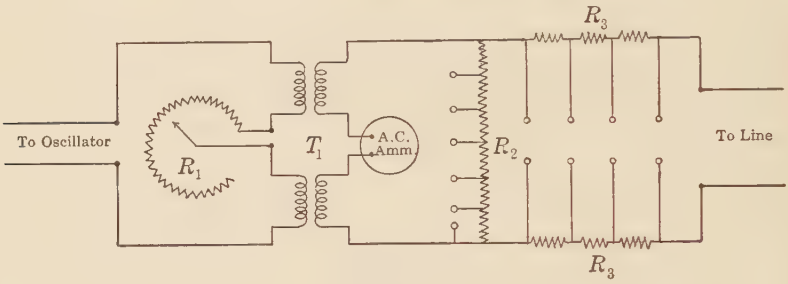

Fig. 90

or open-wire non-loaded lines, (*b*) medium loaded cable lines, and (*c*) heavily loaded cable and open-wire lines, respectively.

The network R_2, connected directly across the secondary of the repeating coil, is made up of three 16-ohm and three 48-ohm rheostats, in series with each other, and individually connected to a switching arrangement (not shown in the figure). By means of this switching arrangement, some of these resistance units may be short-circuited in such a manner that when R_3 is equal to 600 ohms, R_2 is either 16 or 64 ohms; when R_3 is equal to 1200, R_2 is either 32 or 128 ohms, and when R_3 is equal to 1800, R_2 is either 48 or 192 ohms, depending upon the sending-end impedance of the line under test.

Since R_2, under all conditions, is small in comparison with R_3, the current through R_3 and line is very small. Hence the potential across R_3 and the line may be taken as independent of the line impedance connected to the sending element.

A simplified diagram of the receiving element is shown in Fig. 91. It consists of a repeating coil T_2 with three impedance ratios, corresponding to 600, 1200 and 1800 ohms. The primary of this repeat-

ing coil may be connected either to a sending element identical in every respect to the one at the other terminus of the line, or to the terminals of the line under test. The first is done for purposes of calibrating the testing set, and the other for the actual line test, as will be explained below.

In order to calibrate the testing set, the secondary of the repeating coil T_2 is connected across a network E, called a *calibrating network*, in series with rheostat D. For actual line test, the secondary of the repeating coil is connected across a network of three variable resistances, marked in the figure A, B, and C, called an *artificial variable line*, and also in series with rheostat D.

The rheostat D is a circular slide-wire potentiometer, by means of which the potential across the input terminals of a battery of three thermionic amplifiers may be adjusted without changing the current in the *calibrating network* E or in the variable artificial line A, B, C, when either the first or the second is used in the circuit, *i.e.*, either when the set is being calibrated or when it is actually used in line test.

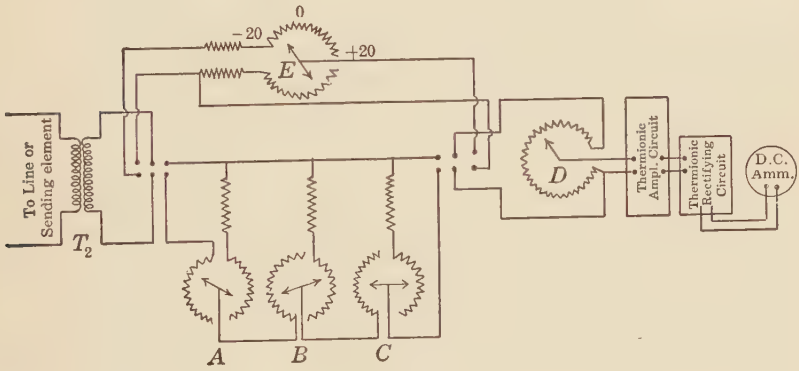

FIG. 91

The potentiometer rheostat D has a maximum resistance of 600 ohms, and is called an *adjustable-dial rheostat.*

A three-electrode thermionic rectifier converts the alternating amplified current output of the battery of amplifiers into direct current to be read on a direct-current ammeter, connected as shown in Fig. 91.

The calibrating network E can be set to give an attenuation equivalent to 21, 41, or 61 standard miles by setting the dial of E

on $-20, 0$, or $+20$ respectively, while the variable artificial line A, B, C may be set to give an attenuation equivalent to a maximum of 41 standard miles, in steps of .1 to a maximum of 1 M. S. C. on dial C, in steps of 1 M. S. C. to a maximum of 5 M. S. C. on dial B and in steps of 5 M. S. C. to a maximum of 35 M. S. C. on dial A.

155. Calibration of the Transmission Measuring Unit. — The calibration of the transmission measuring unit, just previous to the actual measurement of the telephonic transmission equivalent of a line, is done at the terminus of the line where the receiving element is located. Knowing the type of line to be tested, whether loaded or non-loaded, cable or aerial line, resistances R_2 and R_3 of a sending element located at this line terminus are adjusted to be equal approximately to the sending-end impedance of the line under test. The impedance ratio of the repeating coil T_2 is also adjusted to that of the network R_2R_3 of the sending element, which is connected to the primary of T_2.

If the line about to be tested has an equivalent attenuation loss less than 41 M. S. C., the *calibrating dial rheostat* E is set at the position marked 0, which corresponds to an equivalent attenuation produced by 41 M. S. C. If the line about to be tested has an attenuation loss equivalent to between 41 and 61 M. S. C., the dial rheostat E is set in position marked 20, which corresponds to an attenuation equivalent of 61 M. S. C. The reason for so marking the calibrating dial rheostat will appear below.

With the required values of frequency obtained from the oscillator, the secondary sending-element current read on the thermal ammeter, and the receiving-element current read on the d-c. ammeter are adjusted to definite values by means of the rheostat R_1 and the adjustable-dial rheostat D, respectively.

156. Measurement of the Telephonic Transmission Equivalent.— The actual line test is made by disconnecting the primary of the repeating coil T_2 from the sending element used in calibrating, and connecting it, instead, to the terminals of the line under test. A sending element identical to the one used in the calibration of the measuring unit, and with identical network and current adjustments, is connected to the line terminals at the other terminus of the line. The secondary of the repeating coil T_2 is similarly disconnected from the calibrating network E and connected, instead, to the

dial rheostats *A*, *B*, *C* of the artificial variable line. Adjustments are now made by manipulating the dial rheostats *A*, *B*, *C* in such a manner that the line under test, and the artificial variable line, which are now in series with each other, give the same reading on the d-c. ammeter as that obtained during calibration.

It is evident that for the same d-c. current reading, the attenuation produced by the line plus that of the variable artificial line, must be equal to that produced by the calibrating dial *E*. Thus, suppose that the calibrating dial rheostat *E* is set at position 0 for calibration, corresponding to an attenuation of 41 M. S. C., and a certain reading occurs on the d-c. ammeter; and suppose that, for the actual line test, the measuring dials are set to read 10, 3, and 0.4 on *A*, *B*, and *C*, respectively, giving a total reading of 13.4 M. S. C. for the same reading on the d-c. ammeter. Then we shall have

$$\text{transmission equivalent of the line} + 13.4 = 41 \text{ M. S. C.}$$

Hence the transmission equivalent of the line is

$$41 - 13.4 = 27.6 \text{ M. S. C.}$$

On the other hand, suppose that the calibrating dial *E* has been set at position marked 20 for calibration, corresponding to an attenuation equivalent to 61 M. S. C., and that, with the line connected in, and the d-c. current adjusted to a value equal to that for calibration, the measuring dials read a total of 41 M. S. C. Then it is evident that the transmission equivalent of the tested line is $61 - 41 = 20$ M. S. C.

Instead of calculating the telephonic transmission equivalent of the line by subtracting the reading on the measuring dials *A*, *B*, *C* from that for which the calibrating dial has been set for calibration, the measuring dials are calibrated backwards, and are marked in such a manner that they read directly the difference between the attenuation equivalent of the rheostat *E* and that produced by the variable artificial line. The telephonic transmission equivalent may thus be read directly on dials *A*, *B*, *C*.

With the measuring dials *A*, *B*, *C*, calibrated and marked in this manner, it is evident that with the three all "in," resulting in an artificial line attenuation equivalent to 41 M. S. C., the reading would be 0. Hence with the setting of the calibrating dial rheostat *E* at the 0 position, which corresponds to 41 M. S. C., the transmis-

sion equivalent of the line would be 41 − 41 = 0 M. S. C., which is the reading on *A*, *B*, *C*.

Similarly, with the dial rheostats *A*, *B*, *C* all "in," but with the calibrating rheostat *E* set at position marked 20, corresponding to 61 M. S. C., the transmission equivalent of the line under test would be 61 − 41 = 20, which, obviously, is the zero dial reading plus 20.

On the other hand, if the calibrating dial has been set at position marked − 20, and for the line test, the dial rheostats are again all "in," the transmission equivalent of the line under test is 21 − 41 = − 20, which is the zero reading on the dials minus 20. The negative sign, in this particular case, indicates a transmission gain in terms of standard miles.

It follows, from what has been said, that with this backward calibration of the measuring dials *A*, *B*, *C*, the measured transmission equivalent is

Reading on dials *A*, *B*, *C*, minus 20, when dial *E* is set at − 20,
Reading on dials *A*, *B*, *C*, —— when dial *E* is set at 0,
Reading on dials *A*, *B*, *C*, plus 20 when dial *E* is set at 20.

Thus, with a reading on *A*, *B*, *C* of 25, 3, and .5, respectively, giving a total reading of 28.5 M. S. C., the transmission equivalent of the line is

28.5 − 20 = 8.5 M. S. C. with dial *E* set at − 20,
28.5 − 0 = 28.5 " " " " " " 0,
28.5 − 20 = 48.5 " " " " " " 20.

If the readings on *A*, *B*, *C* are 15, 2, and .3, respectively, giving a total reading of 17.3 M. S. C., the transmission equivalent of the line under test is

17.3 − 20 = 2.7 M. S. C. gain with dial *E* set at − 20,
17.3 − 0 = 17.3 " loss " " " " " 0,
17.3 − 20 = 37.3 " loss " " " " " 20.

157. Measurement of the Telephonic Value of a Line. — In the method outlined in the preceding section for the measurement of the transmission equivalent of a line in terms of the standard cable with standard terminations, the only thing considered is the change in volume or loudness due to a change in length of line, or to a

change in termination. All the other factors characteristic of telephonic transmission are entirely ignored.*

These factors are the *pitch*, which depends upon the frequency; the *quality*, which depends upon the component harmonics of the sound wave; the *distortion*, which is due to the disproportionate attenuation and phase shifting of the component harmonics during the period of propagation. While the measured or calculated transmission equivalents, based on attenuation at some definite frequency is intimately related with the speech transmitting qualities of a line, the actual *speech transmitting equivalent* of a line in terms of standard cable must be obtained by an actual comparative speech test.

This is generally done by listening through a standard receiver, alternately connected to the line under test, and to a variable artificial standard cable called a *standard cable set*. It will be noted that three observers are necessary to perform the test. One observer is stationed at the sending end of the line, and repeats uniformly and in an even tone the numbers one to five, into a standard transmitting instrument, connected to the line under test. A second observer is at the receiving terminus of the line, and repeats the same numbers into a transmitter identical with the one at the other end of the line, but connected to one terminal of the standard cable set. A third observer, also stationed at the receiving terminus of the line, listens alternately through a standard receiving instrument that can be switched in, either to the line under test, or to the standard cable set. This last observer adjusts the length of the artificial standard cable until he can hear equally well over the two lines, the standard cable and line under test. Obviously, the length of standard cable that transmits as well as the line under test may be said to be *commercially equivalent* to the line under test. This means that persons speaking normally and with normal hearing can, understandingly, converse with each other equally well over either of the two lines. The line under test may then be said to have a commercial telephonic equivalent of as many standard cable miles as were connected in the circuit for equal speech transmission.

* Present-day practice provides also for attenuation *vs.* frequency tests. These tests enable one to obtain a fairly good idea of the functioning of the line with respect to frequency, and are, therefore, to a certain extent, a measure of those line properties which take into account pitch and distortion.

158. The Loop Test. — The test described in the preceding section depends largely upon the three observers, who are arbitrarily assumed normal speakers and hearers. Moreover, the observer at the sending end of the line under test, and the one at the sending end of the standard cable set, are assumed equivalent to one another with respect to speaking qualities.

This last difficulty may be avoided if the line under test is paralleled with another line similar to it in every respect. If such is the case, the two parallel lines are connected together at the sending end into a single line of double length. Both talker and listener are stationed in separate booths at the receiving terminus of the two lines so connected, and talk alternately over the *looped* lines and the standard cable set. The two looped lines being equivalent or assumed equivalent to one another, the commercial transmission equivalent of each of them is, obviously, one-half the length of standard cable equated to both for equal speech transmission.

If the two parallel lines are known to be dissimilar, the loop test must be preceded or followed by another test to determine the difference in commercial telephonic value between the two lines. For this purpose a speaker at the sending terminus of the two lines talks alternately over the two lines, preferably using the same transmitter. A listener at the receiving end inserts artificial standard cable in series with the line which transmits better, until the two lines have equal transmission values. The length of standard cable, so inserted in series with the better line, is the difference in commercial telephonic value between the two lines in terms of standard miles.

The loop test is then made, as described above, and the commercial telephonic value of the line under test is one-half the length of standard cable equivalent to the lines connected in loop, plus or minus one-half of the difference in telephonic value between the two lines. This *one-half difference* is added when the poorer of the two lines is the one under test, and subtracted when the transmission value of the better line is to be determined.

Thus, if difference in transmission of the two looped lines is found to be 5.00 M. S. C. and the commercial transmission equivalent of the two looped is 26 M. S. C., the transmission equivalent of the better line is $(26–5)/2 = 10.5$ M. S. C., and the poorer line is equivalent commercially to $(26 + 5)/2 = 15.5$ M. S. C.

If three lines are available between any two stations, the commercial transmission equivalents of the three may be obtained by observers stationed at one end of the lines. Thus, if three lines designated by A, B, and C, are available between any two stations, regardless of the routes followed, then by looping lines A and B, B and C, and A and C, three values are obtained from which the transmission equivalents of A, B, and C may be determined. This "*triangulation*," or three-loop measurement, is used extensively in making measurements of line transmission equivalents with the transmission measuring set, as it eliminates the necessity of a set at the distant end of the lines.

159. Disadvantages of the Standard Mile as a Transmission Unit. A Suggested New Transmission Unit. — The results of the analysis given in §§ 148–152 indicate that the transmission equivalents of lines of any length and termination will differ numerically, depending upon whether or not the standard cable in terms of which they are expressed is free from terminal reflections or with receiving terminations similar to those of the line, and also whether or not it is equipped with similar sending-end apparatus. This implies that whenever the transmission equivalent of a line is obtained by actual test or by calculation, the terminations of the standard cable must also be stated, a decided disadvantage in the use of the mile of standard cable as a transmission unit.

As long as the transmission equivalent of a line was measured by an actual comparison of its speech transmitting ability with an actual standard cable of variable length, and equipped with sending-end and receiving-end terminations similar to that of the line, as described in §§ 157–158, the standard mile, though unscientific, was a justifiable unit of transmission.

With the introduction of the transmission measuring set, employing a distortionless artificial circuit (resistance) and single frequency currents for the experimental determination of the transmission ability, or gain or loss in transmission due to change in length or termination, the standard cable as a reference standard of transmission has lost its significance. Yet the mile of standard cable, as a unit of transmission, is still in use, possibly for want of a more suitable one, the adoption of which would materially change present methods of measurement or apparatus used for the purpose.

Since actual length of standard cable is now meaningless as a transmission equivalent in modern test methods, it would be

highly desirable that a transmission unit be formulated and transmission equivalents of lines measured in terms of the *number of these units* instead of in length of standard cable. It would also seem desirable that such a unit should, for reasons stated above, be equal to or nearly equal to that of a mile of standard cable.

It will be recalled that the transmission equivalent of a line with any termination was defined as that length of standard cable free from terminal reflections, whose *nominal admittance* is equal to that of the line under consideration

$$Y_n = \frac{1}{571}\epsilon^{-.107S_c},$$

or

$$\epsilon^{.107S_c} = \frac{1}{571\ Y_n} = \frac{Z_n}{571},$$

where $Z_n = E_s/I_r$ may be called the *nominal impedance* of the line. Solving the preceding equation for S_c, we find

$$S_c = \frac{\log_{10}\dfrac{Z_n}{571}}{.0465}.$$

When S_c is equal to 1, we have

$$\log_{10}\frac{Z_n}{571} = .0465,$$

whence

$$Z_n = 1.113 \times 571 = 635.5 \quad \text{ohms.}$$

It follows, therefore, that a line having a nominal impedance $E_s/I_r = 635.5$ is equivalent to one mile of standard cable. Hence this value of nominal impedance may be chosen as the value of the unit in terms of which transmission values of lines may be either calculated or measured.

Thus for a line whose nominal impedance $E_s/I_r = K\ 635.5$, the transmission value would be

$$\text{Transmission value} = \frac{\log_{10}\dfrac{635.5\ K}{571}}{.0465}$$

$$= \frac{\log_{10}\dfrac{635.5}{571}}{.0465} + \frac{\log_{10}K}{.0465}$$

$$= 1 + \frac{\log_{10}K}{.0465}\ \text{units.} \tag{14}$$

Using this expression, the calculation of the transmitting value of a line, once its nominal admittance or impedance is known, becomes a very simple matter.

Suppose, for instance, that the nominal impedance of a line, at the average telephonic frequency of 796 cycles per second, is found by actual measurement to be $E_s/I_r = 1271$ ohms. Then we have

$$K = \frac{1271}{635.5} = 2;$$

hence the transmission value of the line is

$$\text{Transmission value} = 1 + \frac{\log_{10} 2}{.0465} = 1 + \frac{.30103}{.0465}$$

$$= 1 + 6.47 = 7.47 \quad \text{units of transmission.}$$

Similarly, the nominal impedance of a 30-mile No. 19 non-loaded cable, terminated with a receiver of 300 $\epsilon^{j50°}$ ohms, as calculated in § 121, is

$$Z_n = \frac{616}{.0808} = 7623.7 \quad \text{ohms}$$

whence

$$K = \frac{7623.7}{635.5} = 11.99,$$

and the transmission value is

$$\text{Transmission value} = 1 + \frac{\log_{10} 11.99}{.0465} = 1 + \frac{1.078819}{.0465}$$

$$= 1 + 23.2 = 24.2 \quad \text{units of transmission.}$$

160. Proposed A. T. & T. Transmission Unit. — Since the writing of the preceding section, the author has been informed that the American Telephone and Telegraph Company has been contemplating for some time the adoption of a transmission unit to replace the mile of standard cable as a measure of telephonic transmission. The following memoranda giving the arguments for the proposed change and the description of the new unit is included through the courtesy of Mr. B. D. Hull, General Transmission and Protection Engineer, Southwestern Bell Telephone Company:

> "On account of the characteristics of the standard cable, different frequency components of the voice currents transmitted over it are attenuated by different amounts. In addition, therefore, to the volume reduction, there is also a distorting

effect, introduced by the standard cable. This distortion was, however, comparable to the distortion introduced by most commercial circuits at the time it was first used, and for that reason was suitable for use as a standard of reference. With the increased proportion of high quality circuits, and the general use of single frequencies for transmission measurements instead of voice currents, the standard cable has become somewhat unsatisfactory as a unit of transmission. Also the increasing use of carrier and radio, in which the mile of standard cable has no significance, is another argument for a distortionless unit.

"For transmission testing work where the single-frequency method is employed, the use of a distortionless artificial line has become prevalent in the last few years, and has proven very satisfactory for this purpose. All field testing sets are now made up with a distortionless variable line. The unit which has been selected for this purpose has an attenuation of 0.109, corresponding to that of the standard cable at 796 cycles. This particular figure was taken on the basis that it represented the average attenuation of talking currents in standard cable. Recent tests have shown, however, that with present standard substation apparatus, the average attenuation of standard cable for voice currents is approximately 0.122, which corresponds to the attenuation attained with a single frequency of 1000 cycles.

"Proposed Unit. The present proposals which have been agreed upon by the various departments of the American Telephone and Telegraph Company and the Western Electric Company are as follows:

"1. — That a distortionless transmission unit be standardized as the unit of difference in transmission volume.

"2. — That a power ratio of $10^{0.1}$ be selected as this unit. This power ratio corresponds to a current ratio of $10^{0.05}$ or $\epsilon^{0.115}$.

"3. — That the new unit be called for the present merely 'Transmission unit,' abbreviated TU. This may be replaced later by a name, if generally agreed upon.

"4. — That a reproducible and approximately distortionless transmission reference system be established to replace the present system reference standard.

"5. — That the transmission reference system so established be adjusted at 24 units to give equivalent voice to ear efficiency equal to that of the present system reference standard adjusted to 24 miles of standard cable.

"Advantages of the Proposed Unit. 1. — The proposed unit is most convenient from the standpoint of computation, as it is based on the common logarithmic base, and for this reason, the handy Briggs or common log. table can be used to calculate the current ratios. Ten units correspond to a power ratio of 10, and intermediate units correspond very closely to the power ratios shown below, which are easily remembered:

No. of Units	1	2	3	4	5	6	7	8	9	10
Power Ratio	1.25	1.6	2.	2.5	3.2	4.	5.	6.4	8.	10.

"The fact that these figures are easily remembered makes mental conversion between transmission units and power ratio very easy. Only one series of numbers need to be memorized, as the ratios for units from 10 to 20 are the same as those given in the above table, with the decimal point set over one place. For example, 16 units correspond to a power ratio of 40, etc., and thus the power ratio corresponding to any number of units is readily visualized.

"2. — The series of numbers represented above are coming into general use in several countries of Europe, as so-called 'preferred numbers' for standard sizes, and are being considered for use in this country.

"3. — It is suitable for international standardization as it is a distortionless unit of convenient magnitude.

"4. — The value of $10^{\cdot 1}$ power ratio, current ratio $\epsilon^{\cdot 115}$, is midway between the two values which we are now using, namely $\epsilon^{\cdot 109}$, which is used for transmission testing work and $\epsilon^{\cdot 122}$, which is the average equivalent attenuation of voice currents in standard cable.

"5. — The attenuation of the proposed unit is near enough to that of the unit now used in testing apparatus so that the expense and inconvenience of changing the testing equipment will be small. Certain types of testing equipment will not be modified, while others can be modified in the field whenever the companies feel that this is desirable. Meanwhile unconverted appa-

ratus can be used for a large part of the work without correcting the results, because the error involved in so doing would be only 5 per cent. If it is desired to change the sets, it is thought that this can be done by sending out suitable resistances to be substituted for those now in the artificial lines by the engineer on the job. If handled in this way the cost to the entire Bell System for new resistances should not exceed $7000.

"Transmission Standards on the New Basis. It should be noted that the requirements mentioned in recommendation 5 are of considerable importance from a practical standpoint, as the transmission standards now in use throughout the country are so fixed in the minds of the executives and the transmission engineers that it would be very undesirable to require a change. If the new system reference circuit is set up as recommended in recommendation 5, the talk given over a circuit of 24 units will correspond with that over 24 miles of standard cable, and as this figure represents an approximate average of standards now in use throughout the plant, the standards can probably remain unchanged. The relation between the figures for the other standards and miles are as follows:

Transmission Units	18	24	30
Miles Standard Cable	18.3	24.0	29.7

"Another interesting comparison in order to show the difference between the two units is to compare the two units in terms of attenuation of standard cable. The old transmission unit $\epsilon^{.109}$ is equal to the attenuation of standard cable for a frequency of 796 cycles. The equivalent voice frequency attenuation of standard cable as now estimated, namely, $\epsilon^{.122}$, is equal to the attenuation for a frequency of 1000 cycles. The proposed unit $\epsilon^{.115}$ is equal to the attenuation of standard cable at 890 cycles.

"The comparison given above may be very useful in explaining the significance of the new unit to people not closely connected with transmission work. It can be pointed out that the new unit, like the units now in use, corresponds to the effect of a mile of standard cable on currents of telephonic frequency. The change in magnitude as compared with the unit of $\epsilon^{.109}$ now used in testing apparatus, is equivalent to slightly changing

the frequency of the current used for referring transmission losses to the effect of standard cable."

161. Discussion of the Proposed New A. T. & T. Unit. — The method involved in the calculation of line transmitting values in terms of the A. T. & T. transmission unit is practically identical with that involved in terms of the unit suggested in (14), § 159. Barring the difference in attenuation constants, .115 hyperbolic radians in one, and .107 in the other unit, the transmission unit suggested in § 159 presupposes equal sending-end voltages, *i.e.*, identical transmitting apparatus, while the A. T. & T. unit presupposes equal sending-end currents, seemingly neglecting the effect of the sending-end impedance.

To get at the method of calculating line transmission values in terms of the A. T. & T. transmission unit, consider a line of length S_c, having an attenuation constant of .115 hyperbolic radians, and terminated with an impedance equal to its characteristic impedance, so that the current decays exponentially. If I_s and I_r are the sending-end and receiving-end currents, respectively, then

$$\frac{I_s}{I_r} = \epsilon^{.115 S_c},$$

whence

$$S_c = \frac{\log_{10} \frac{I_s}{I_r}}{.115 \log_{10} \epsilon} = \frac{\log_{10} \frac{I_s}{I_r}}{.05}.$$

If S_c is equal to 1, then

$$\log_{10} \frac{I_s}{I_r} = .05,$$

or

$$\frac{I_s}{I_r} = 1.122$$

Referring this to the proposed A. T. & T. transmission unit, this line, or any other line of any length and characteristics whatever, whose current ratio is 1.122, will have a transmission value of one TU. Any other line presenting a current ratio of

$$\frac{I_s}{I_r} = 1.122\, K$$

will, therefore, have a transmission value of

$$(15) \quad \frac{\log_{10} 1.122\, K}{.05} = \frac{\log_{10} 1.122}{.05} + \frac{\log_{10} K}{.05} = 1 + \frac{\log_{10} K}{.05} \; TU.$$

Suppose, for instance, that the current ratio of a line, obtained either by calculation or test, is found to be equal to 6.732, then

$$K = \frac{6.732}{1.122} = 6.$$

By equation (15), the transmission value of the line would be

$$1 + \frac{\log_{10} 6}{.05} = 1 + \frac{0.778151}{.05} = 16.563\ TU.$$

Similarly, if the transmission value of a line, obtained by test, is found to be 10 TU, then

$$10 = 1 + \frac{\log_{10} K}{.05},$$

whence $\log_{10} K = .45$, or $K = 2.819$, and since

$$\frac{I_s}{I_r} = 1.122\ K,$$

it follows that

$$\frac{I_s}{I_r} = 3.163$$

Also, since

$$\frac{I_s}{I_r} = \epsilon^{aS}$$

by taking logarithms, or from tables of exponential functions, we get

$$aS = 1.152 \text{ hyperbolic radians.}$$

If the line under test is known to be a No. 19 B. & S. cable, having an attenuation constant of .107 hyperbolic radians at the same frequency as the unit, then the length of the line is

$$S = \frac{1.152}{.107} = 10.7 \text{ miles.}$$

If the line under test is known to be a No. 10 B.& S. open wire line, having an attenuation constant of .009119 at the same frequency as the unit, the length of the line is

$$S = \frac{1.152}{.009119} = 120.3 \text{ miles.}$$

Similarly, if the No. 10 B. & S. were loaded with impedance coils of $1000/89.5^{\circ}$ ohms at a spacing distance of 9.64 miles, and having an

attenuation constant of .004359 hyperbolic radians (Table 9), the length of the line would be

$$S = \frac{1.152}{.004359} = 264.2 \text{ miles.}$$

These three lines of different characteristics and different lengths have the same transmission value in terms of the proposed new A. T. & T. transmission unit. *Are they also equivalent to one another under identical sending-end and receiving-end terminations?*

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LIST OF SYMBOLS

a	Attenuation constant of any line in real hyperbolic radians.
a_i	Attenuation constant of an ideally loaded line.
a_l	Attentuation constant of a regularly series-loaded line.
a_o	Attenuation constant of a non-loaded line.
a_q	Attentuation constant of a uniform line equivalent to a regularly series-loaded line.
c	Capitance per mile of line conductor in farads.
C	Capacitance per loop mile or per mile of line.
C_o	Capacitance per mile of line before loading.
C_t	Total capacitance of any line.
d	Diameter of line conductor; also subscript to indicate forward or direct electromagnetic wave.
$\mathcal{D}$	Inside diameter of lead sheath cable.
D	Interaxial distance between line conductors.
e	Instantaneous voltage.
E	Tensor or vector voltage at any point along a line.
E_b	Battery voltage in plate-filament circuit of a thermionic tube.
E_d	Voltage at the end of a line due to a forward or direct electromagnetic wave.
E_f	Voltage across the filament of a thermionic tube.
e_g	Instantaneous grid-filament potential of a thermionic tube.
E_g	Constant grid-filament voltage of a thermionic tube.
E_{ga}	Amplitude of a sinusoidal voltage superimposed upon the grid-filament potential.
E_{gaf}	Effective value of grid-filament voltage.

e_p	Instantaneous values of alternating component of plate-filament potential.
E_p	Plate-filament voltage.
E_{pa}	Amplitude of alternating component of plate-filament potential.
E_{rf}	Effective voltage value across receiver in plate-filament circuit.
E_r	Tensor or vector voltage at the receiving end of a line.
E_ρ	Reflected voltage at the end of a line due to a reflected electromagnetic wave.
E_s	Vector or tensor voltage at the sending end of a line.
E_{sl}	Vector or tensor voltage at the sending end of a regularly series-loaded line.
f	Frequency in cycles per second.
g	Leakage conductance per mile of line conductor in mhos.
G	Leakage conductance per loop mile or mile of line.
G_o	Leakage conductance per mile of line before loading.
y	Distance of line conductor from the ground.
i	Instantaneous current.
I	Tensor or vector current at any point along a line.
$I_\imath$	Current at the end of a line due to a forward or direct electromagnetic wave.
I_p	Plate-filament current in a thermionic tube.
I_{pa}	Amplitude of alternating component of plate-filament current.
I_{paf}	Effective value of the alternating component or the plate-filament current.
I_r	Tensor or vector current at the receiving end of a line.
I_ρ	Reflected current at the end of a line due to a reflected electromagnetic wave.
I_s	Tensor or vector current at the sending end of a line.
I_{sl}	Tensor or vector current at the sending end of a regularly series-loaded line.

j Vector operator $\sqrt{-1}$.

k Specific inductive capacity of a dielectric medium.
k_{re} Voltage transition coefficient.
$k_{\rho e}$ Voltage reflection coefficient.
k_{rei} Voltampere transition coefficient.
$k_{\rho ei}$ Voltampere reflection coefficient.
k_{ri} Current transition coefficient.
$k_{\rho i}$ Current reflection coefficient.
k_{rw} Watts transition coefficient.
$k_{\rho w}$ Watts loss coefficient.

l Inductance per mile of line conductor in henries.
L Inductance per loop mile or per mile of line.
L_l Inductance of loading coil.
L_0 Inductance per mile of line before loading.
L_t Total inductance of a line.

$M.\ S.\ C.$ Miles of Standard Cable.

n Any integer; also ratio of turns of repeating coil.
N Subscript to indicate nominal artificial circuit or line nominally equivalent to a loaded line.
N_1, N_2 Constants; also primary and secondary turns, respectively, of a repeating coil.
N_N Number of loading coils per wave-length.

p Tensor or vector propagation constant of any line.
p_c Tensor or vector propagation constant of standard cable.
p_i Tensor or vector propagation constant of an ideally loaded line.
p_N Tensor or vector propagation constant of a line nominally equivalent to the regularly series-loaded line.
p_0 Tensor or vector propagation constant of a line before loading.
p_q Tensor or vector propagation constant of uniform line equivalent to the regularly series-loaded line.

P	Power in watts.
P_R	Dissipative power component of vector voltamperes.
P_{rm}	Maximum dissipative power in receiving-end impedance of a line.
P_X	Reactive power component of vector voltamperes.
q	Electric charge. Quantity of electricity. Subscript to indicate uniform line equivalent to a regularly series-loaded line.
r	Any resistance in ohms. Also resistance per mile of line conductor.
R	Any resistance. Also resistance per loop mile or per mile of line.
R_l	Effective resistance of a loading coil.
R_0	Resistance per mile of line before loading.
R_p	Internal resistance of a thermionic tube.
S	Length of a line in miles.
S_c	Length of standard cable equivalent in transmission to any line.
S_d	Spacing distance between loading coils.
$S.\ M.$	The standard mile; one mile of standard cable.
S_r	Distance along a line measured from the receiving end.
S_s	Distance along a line measured from the sending end.
t	Time in seconds.
T	Period; time in seconds per cycle; also subscript to indicate artificial T circuit.
TU	The Bell Transmission Unit.
u	Real hyperbolic angle in hyperbolic radians.
v	Wave-length constant or phase constant of any line in circular radians.
V	Velocity of propagation of an electromagnetic wave in miles per second.

v_i Wave-length constant of an ideally loaded line.

v_l Wave-length constant of a regularly series-loaded line.

v_N Wave-length constant of uniform line nominally equivalent to a regularly series-loaded line.

v_o Wave-length constant of a line before loading.

v_q Wave-length constant of uniform line equivalent to a regularly series-loaded line.

(VA) Tensor or vector voltamperes.

$(VA)_m$ Maximum voltamperes.

$(VA)_{mu}$ Maximum useful amplified voltamperes obtainable through a thermionic tube.

$(VA)_r$ Tensor or vector voltamperes at the receiving end of a line.

$(VA)_s$ Tensor or vector voltamperes at the sending end of a line.

x Instantaneous value of a sine wave; also adjusted reactance component of the balancing impedance of Wheatstone bridge.

X Amplitude of a sine wave. Any reactance. Also subscript to indicate reactive power.

y Instantaneous value of any harmonic function. Also vector linear line admittance $G + jC\omega$.

Y Any admittance.

Y_n Nominal line admittance, i.e., ratio of receiving-end current to sending-end voltage.

Y_{nc} Nominal line admittance of any length of standard cable.

Y_m Mutual admittance of a thermionic tube.

Y_o Characteristic line admittance.

Y_p Internal admittance of a thermionic tube.

z Vector linear impedance $R + jL\omega$.

Z Tensor or vector impedance at any point along a line. Also any impedance.

Z_{io} Tensor or vector characteristic impedance of an ideally loaded line.

Z_l Tensor or vector impedance of a loading coil.

Z_L Measured line impedance.

Z_N Tensor or vector impedance of artificial T circuit nominally equivalent to a line.

Z_{N1}, Z_{N2} Branch impedances of the artificial T circuit nominally equivalent to a line.

Z_m Mutual impedance of a thermionic tube.

Z_0 Tensor or vector characteristic impedance of any line; Impedance of any line.

Z_{01}, Z_{02}, Z_{03} Tensor or vector characteristic impedances of sections of a composite line.

Z_{0c} Tensor of vector characteristic impedance of the standard cable.

Z_{0l} Characteristic impedance of a loaded line.

Z_p Internal impedance of a thermionic tube.

Z_{q0} Tensor or vector characteristic impedance of uniform line equivalent to the regularly series-loaded line.

Z_r Tensor or vector impedance of receiving-end apparatus.

Z_{rm} Receiving-end impedance for maximum voltamperes.

Z_s Tensor or vector impedance at the sending end of a line.

Z_{sl} Tensor or vector sending-end impedance of a loaded line.

Z_T Tensor or vector impedance of the artificial T circuit of a line.

Z_π Tensor or vector impedance of the artificial II circuit of a line.

Z_1, Z_2 Branch impedances of the artificial T circuit equivalent to any line; also impedance branch of Wheatstone bridge.

α Characteristic phase angle of a line in degrees or circular radians.

α_1 Phase angle of linear impedance vector z.

α_2 Phase angle of linear admittance vector y.

α_N Characteristic phase angle of uniform line nominal to regularly series-loaded line.

α_q Characteristic phase angle of uniform line equivalent to the regularly series-loaded line.

$\beta_1, \beta_2, \beta_3, \beta_4$ Phase angles of vector hyperbolic functions of complex angles.

δ Angle of propagation-constant vector of any line.

δ_i Angle of propagation-constant vector of an ideally loaded line.

δ_N Angle of propagation-constant vector of uniform line nominally equivalent to the regularly series-loaded line.

δ_q Angle of propagation-constant vector of uniform line equivalent to the regularly series-loaded line.

ϵ Base of Naperian logarithms.

θ Any angle. Phase angle between voltage and current vectors at any point along a line.

θ_l Phase angle of loading-coil impedance vector.

θ_r Phase angle of receiving-end impedance vector.

θ_s Phase angle between voltage and current at the sending end of a line.

λ Length of electromagnetic wave along any line.

λ_i Length of electromagnetic wave along an ideally loaded line.

λ_N Length of electromagnetic wave along a uniform line nominally equivalent to the regularly series-loaded line.

λ_q Length of electromagnetic wave along a uniform line equivalent to the regularly series-loaded line.

μ Magnetic permeability. Amplification factor of a thermionic tube.

μ_1 Useful voltage amplification factor of a thermionic tube.

ρ Subscript to indicate reflected voltage, current or volt-amperes.

ϕ Any phase angle; magnetic flux.

ϕ_d Angular spacing distance in degrees or circular radians.

ϕ_r Phase angle between receiving-end and sending-end voltages.

ω Angular velocity in circular radians per second.

INDEX

Ability, transmission, 257
Adjustable dial rheostat, 271
Admittance, internal, 208
  linear line, 25
  mutual, 211
  nominal, 261
Amplification, 203
  current, 227
  factor, 214
  One-way, 225
  power, 216
  parallel-series, 231
  series-parallel, 228
  useful, 230
  voltage, 216
Amplified voltage, 216
Amplifiers, 203
Amplifier action, 211
Amplifying circuit, 225
  type "21", 233
  type "22", 233
Amplifying tube, standard, 217
Amplitude, 5
Angular spacing distance, 188
Angular velocity, 5
Aram Boyajian, 20
Arithmetic mean, 6
Artificial line, 255
Artificial testing standard, 266
Artificial variable line, 271
Attenuation constant, 33
  measurement of, 244
  scalar value, 35
Attenuation factor, 34
A. T. & T. Transmission unit, 279
  discussion of, 283
Average value, 6

Balancing network, 233
Balancing terminal network, 255
Basic network, 255
Bibliography, 287
Boyajian, Aram, 20

Bridge, impedance, 250
Bridge, Wheatstone, 242

Cable set, standard, 275
Cable, standard, 257
Capacitance, aerial line, 174
  concentrated, 55
  line, 51
  single overhead conductor, 174
Calibrating dial rheostat, 272
  network, 271
Characteristic impedance, 36
  nominal line, 185
  series loaded line, 182
  smooth line, 236
  variation of, 163, 165, 166
Characteristic phase angle, 37
  measurement of, 242
Circuit, amplifying, 225
  equivalent, 139
  nominal, 140
  repeater, 225
Coil, loading, 178
Coil, terminal repeating, 199
Commercial equivalent, 275
Complex functions, 9
  angles, hyperbolic functions of, 22
  number, 9
  quantities, trigonometric functions of, 21
Complex wave analysis, 7
Composite line, 96, 201
Concentrated capacitance, 55
Concentrated inductance, 55
Conductance, line, 25
  variation of, 175
Conductivity, unilateral, 208
Consonantal sounds, 2
Cosine function, 5
Current along infinite line, 44
  loaded line, 95
  open-circuited line, 62
  short-circuited line, 81

Current at short-circuited end, 79
in receiving-end impedance, 94
Current, sending-end, loaded line, 93
open-circuited line, 48
Current, plate, of thermionic tube, 206
Current, pulsating, 209
Current reflection coefficient, 113
Current, saturation, of thermionic tube, 207
Current transition coefficient, 113
Cycle, 5

DeForest, 209
Dissipative power, 184
Distortion, 167, 275
Distortionless line, 167, 173
Distortionless transmission, 171

Edison effect, 206
Edison, Thomas, 206
Effective value, 6
Electric constants
No. 10, B. & S. line, 38
telephone lines, 40
Electric wave propagation, 25
Electromagnetic waves, 102
Electron, 204
Electronic emission, 204
Element, receiving, 269
sending, 269
Emission, thermionic, 204
Energy, 204
Energy transfer, 101, 196
Energy gain due to loading, 197
Equivalence of
loaded line and infinite line, 92
nominal and smooth line, 186, 187
Equivalent circuit, 139
Equivalent circuit of series loaded line, 179
Equivalent H circuit, 140
Equivalent I circuit, 140
Π circuit, 156, 141
T circuit, 141, 143
Equivalent line capacitance, 55
inductance, 55
Equivalent Π of loaded line, 162
of open-circuited line, 160
of short-circuited line, 160
Equivalent T of loaded line, 153
of open-circuited line, 150
of short-circuited line, 152
Equivalent transmission, 258, 262
Equivalent telephonic transmission, 266
Equivalent, uniform line, 170
Exponential equivalents
hyperbolic functions, 19
hyperb. funct. of complex numbers, 24
trigonometric functions, 11
trigonometric funct. of complex numbers, 24

Ferranti effect, 56
Frequency interval, 247
Frequency, natural, 50
Frequency, resonant, 54
Fleming, J. A., 206
Fleming valve, 206, 208

Gain, terminal, 126, 128
General transmission equations, 31
interpretation of, 31
Gherardi, B, 203
Grid, thermionic tube, 209
Grid potential, 209

H, Equivalent circuit, 140
Harmonic components, 7, 9
Harmonic function, 2, 5
Hull, B. D., 279
Hyperbola, rectangular, 16
Hyperbolic angle, 18
cosine, 15
functions, 15
Hyperbolic functions
of complex quantities, 22
exponential equivalents of, 19
relations between, 15, 17, 18
Hyperbolic radian, 17
sine, 15
tangent, 15

Imaginary angles
hyperbolic functions of, 21
trigonometric functions of, 20
Impedance along
loaded lines, 95
open-circuited lines, 65
short-circuited lines, 84
Impedance bridge, 250
Impedance, characteristic, 36
Impedance-frequency relation, 236
Impedance of infinite line, 43
Impedance, internal, 208

Impedance, line, 235
linear line, 25
mutual, 211
Impedance, sending-end
of composite line, 96
of loaded line, 90
of open-circuited line, 47
of short-circuited line, 77
Impedance, surge, 37, 103
Inductance, concentrated, 55
line, 51
Inductance-frequency curve, 253
Inductance, of loading coil, 202
Inductometer, 250
Infinite line, 43, 114
Internal admittance, 208
Internal impedance, 208
Instantaneous value, 5
Irregularity of line, 237
location of, 247

Jewett, F. B., 203

Krarup system, 177

Leak loading, 178
Line, artificial, 255
Line calculation data
No. 10, B. & S. line, 41
Line, composite, 96
Line constants
variation of, 174
Line conductance, 25
measurement of, 246
Line distortion, 173
Line impedance, 235
measurement of, 241, 248
Line inductance, 51
variation of, 176
Line irregularities, 237
effect of, 238
location of, 247
Line, regularly series-loaded, 179
Line resistance, 51
measurement of, 246
variation of, 175
Linear line admittance, 25
Linear line impedance, 28
Loading coil, 178
Loading coil inductance, 202
Loading coils
number per wave-length, 184
spacing distance between, 188
Loading, concentrated, 177
Loaded line
equivalent Π 162
equivalent T of, 153
regularly series, 178
with end receiver, 90
Loaded oscillator, 220
Loading, leak, 178
lumped, 177
of underground cables, 201
parallel, 178
series, 179
uniform, 177
Loop test, 276
Loss coefficient, watts, 138
Loss, terminal, 126, 128
Loudness of sound, 2
Lumped loading, 177

Mathematical equivalent of lines, 147, 148
Measuring set, transmission, 266
Measuring unit, transmission, 269
Microphone, 1
Mile, standard, 257
Morecroft, J. H., 204
Mutual admittance, 211
Mutual impedance, 211

Natural frequency, 50
Natural impedance, 37, 103
Network, balancing, 233, 255
Network, basic, 255
Network, calibrating, 271
Nominal admittance, 261
Nominal circuit, 140
Nominal H circuit, 140
Nominal I circuit, 140
Nominal Π circuit, 141
Nominal T circuit, 141
Nominal uniform line, 184

One-way amplification, 225
Open-circuited line, 37
equivalent Π of, 160
equivalent T, 150
Oscillator, 210, 217
loaded, 220
Oscillations, self-sustained, 219

Parallel loading, 178
Period, 5
Phase, 6

Phase angle
along open-circuited line, 69
along short-circuited line, 86
at open end of a line, 71
Phase angle, characteristic, 37
Phase constant, 34, 44
Phase factor, 34
Pitch, 2, 275
Plate current, thermionic tube, 205
Power, amplification, 216, 226
Power dissipative, 9, 114
reactive, 9, 114
Power relations, 114
Propagation constant, 32
measurement of, 244
of nominal line, 185
series loaded line, 182
variation of, 167, 169, 170
Propagation factor, 182
Pupin, M. I., 178
Pupin system, 177
Pulsating current, 209

Quality of sound, 2, 275

Radian, circular, 5, 17
hyperbolic, 17
Reactive power, 9, 114
Receiver impedance,
variation of, 120
Receiving-end current, 94
Receiving-end voltage, 94
Receiving element, 269
Rectangular hyperbola, 16
Reflection phenomena, 105
at end of loaded lines, 110
on open-circuit, 105
on short-circuit, 108
Reflection coefficient,
current, 113
voltage, 112
voltamperes, 133
Regularly series-loaded line, 179
Relay, 203
Repeaters, 203
Repeater action, 204
Repeater circuit, 225
Repeating coil, 198
Repeater, Shreeve, 203
Resistance-frequency, relations, 253
Resonance, 50, 54
Rheostat, adjustable dial-, 271
calibrating dial-, 272
Root mean square, 6

Saturation curve, thermionic tube, 207
Scalar value
attenuation constant, 35
phase constant, 35
wave-length constant, 35
Self-sustained oscillations, 219
Sending element, 269
Sending-end current
composite line, 96
infinite line, 43
loaded line, 93
open-circuited line, 48
short-circuited line, 78
Sending-end impedance
composite line, 96
infinite line, 43
loaded line, 90
open-circuited line, 47
short-circuited line, 77
Sending-end voltamperes, 115
Series equivalents
trigonometric functions, 11
Series loading, 179
Series-parallel amplification, 228
Short-circuited line, 77
equivalent Π, 161
equivalent T, 152
Shreeve repeater, 203
Sine current, 6
Sine curve, 5
Sine electromotive force, 6
Sine function, 2
Smooth line, impedance of, 236
Sound waves, 2
Spacing distance, 184
Spacing distance, angular, 188
Spacing distance, calculation of, 190
Spacing distance, maximum, 188
Speech transmission equivalent, 275
Standard amplifying tube, 217
Standard cable, 257, 259
Standard cable set, 275
Standard mile, 257
Standard transmission, 282
Standing waves, 102
Surge impedance, 37, 103
Symbols, list of, 291

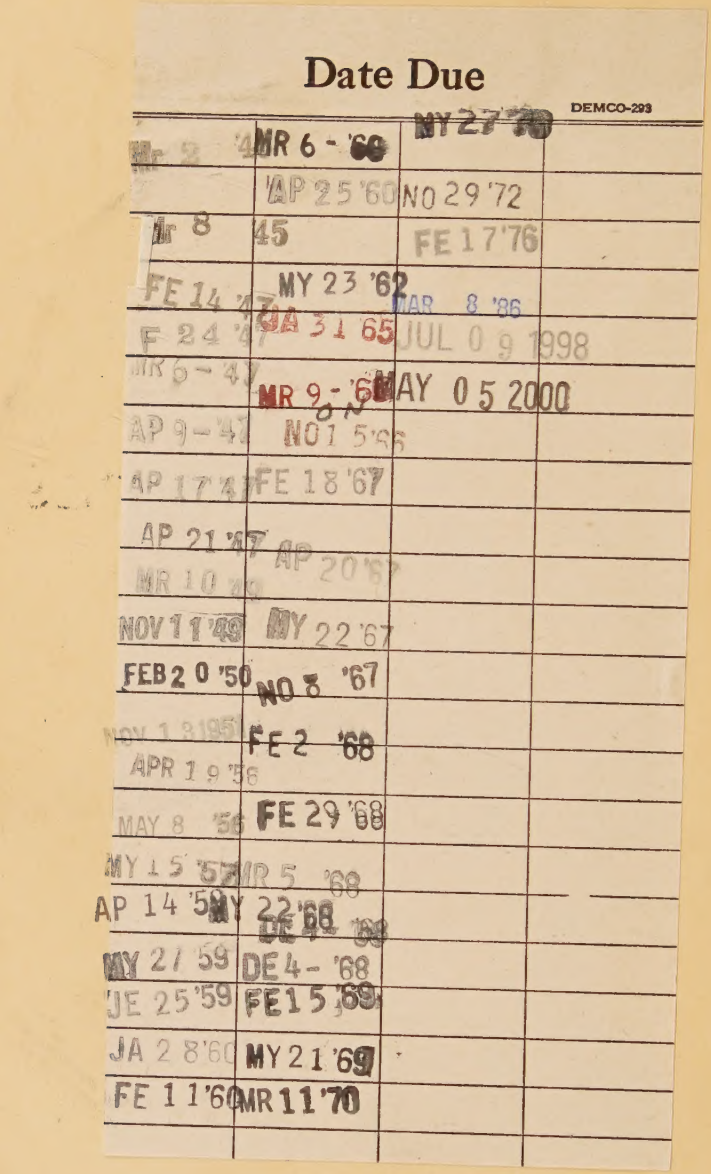
Telephonic transmission, equivalent, 266
Telephonic value, 257, 269
measurement of, 274
Terminal gain, 126, 128

Terminal loss, 126, 128
Terminal network, 255
Terminal repeating coil, 199
Test, loop, 276
Testing standard, artificial, 266
Third electrode of thermionic tube, 209
Thermionics, 204
Three-electrode tube, 209
Thermionic emission, 204
Thermionic tube, 204
Timbre, 2
Transition coefficient,
  current, 113
  voltage, 111
  voltampere, 132
  watts, 136
Transmission ability, 257
Transmission constants,
  No. 10 B. & S. line, 39
  table of, 40
Transmission, distortionless, 171
Transmission equations, 31
Transmission equivalents, 258, 262
Transmission equivalent, commercial, 275
Transmission equivalents,
  measurement of, 269, 272
  of lines, 260
Transmission line, elements of, 25
Transmission measuring set, 269
Transmission measuring unit, 269
Transmission standards, 282
Transmission unit, 277
Transmission unit, A. T. & T., 279
Transmitting equivalent, speech, 275
Transmitter, 1
Traveling waves, 103, 104
Triangulation, 276
Trigonometric functions, 5
  of complex quantities, 21,
  of imaginary angles, 20
Trigonometric relations, 14
TU, 280
TU, advantages of, 281
Tube, thermionic, 204
Two-electrode tube, 206
Two-way amplification, 233, 237

Underground cables,
  loading of, 201
  properties of, 201
Uniform loading, 177
Unilateral conductivity, 207
Unit, A. T. & T. transmission, 279
Unit, transmission, 277
Useful amplification, 230

Vacuum tube, 204
Value, telephonic, 257, 269
Variable line, artificial, 271
Vector, 5
  addition, 10, 12
  evolution, 13
  differentiation, 13
  integration, 13
  involution, 13
  operations, 10
  subtraction, 12
Velocity, angular, 5
Velocity of propagation, 35, 51
Van Der Bijl, H. J., 204
Voltage amplification, 227
Voltage amplification factor, 216
Voltage along
  end-loaded line, 95
  infinite line, 44
  open-circuited line, 59
  short-circuited line, 80
Voltage at
  open-circuited end of a line, 48
  at receiving-end of line, 94
Voltage reflection coefficient, 112
Voltage transition coefficient, 111
Voltamperes, 114
Voltamperes reflection coefficient, 132
Voltamperes transition coefficient, 132

Watts, 114
Watts loss coefficient, 136, 138
Watts transition coefficient, 136, 138
Wave-length, 34
  calculation of, 34
Wave-length constant, 34
  scalar value, 36
Wave-length factor, 34
Wave, electromagnetic, 102
Wave, standing, 102
Wave, propagation of an electromagnetic, 25
Wave, traveling, 103
Wheatstone, bridge, 242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DE 4 - '68
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MAY 05 2000

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